NSW HSC 4 Unit Mathematics Examination 1979

Question 1.

Let

$$f(x) = x^3 + px + q = (x-\alpha)(x-\beta)(x-\gamma),$$

where p, q are real and α, β, γ are (not necessarily distinct) real numbers.

(i) If $f(\alpha) = f'(\alpha) = 0$, prove that $(x-\alpha)^2$ is a divisor of $f(x)$.

(ii) If $\alpha = \beta, \alpha \neq \gamma$, so that $f(x) = (x-\alpha)^2(x-\gamma)$, prove that

(a) $p \neq 0$;

(b) $4p^3 + 27q^2 = 0$;

(c) $\alpha = \frac{-3q}{2p}$.

(iii) Solve completely the equation $x^3 - 27x - 54 = 0$.

Question 2.

Given that the highest common factor d of two integers a, b is expressible as

$$d = al + bm$$

where l, m are integers, prove that:

(i) if c, d are integers, p is a prime, and p divides cd, then p divides at least one of c and d;

(ii) the equation

$$ax + by = c,$$

where a, b, c are given integers, has integer solutions x, y if and only if the highest common factor of a and b is a divisor of c;

(iii) if $\frac{a}{b}$ is a rational number in reduced form, then there exist infinitely many distinct rational numbers $\frac{r}{s}$ such that

$$|\frac{a}{b} - \frac{r}{s}| \leq \frac{1}{s}.$$

Question 3.

Prove that the range on a horizontal plane of a particle projected upwards at an angle α to the plane with velocity V metres per second is $\frac{V^2 \sin 2\alpha}{g}$ metres, wher g metres per second per second is the acceleration due to gravity.

A garden sprinkler sprays water symmetrically about its vertical axis at a constant speed of V metres per second. The initial direction of the spray varies continuously between angles of 15° and 60° to the horizontal.

Prove that, from a fixed position O on level ground, the sprinkler will wet the surface of an annular region with centre O and with internal and external radii $\frac{V^2}{2g}$ metres and $\frac{V^2}{g}$ metres respectively.

Deduce that by locating the sprinkler appropriately relative to a rectangular garden bed of size 6 metres by 3 metres, the entire bed may be watered provided that

$$\frac{V^2}{2g} \geq 1 + \sqrt{7}.$$

Question 4.

(i) Write down the binomial expansion of $(1+x)^n$ (n a positive integer).

As the result of an experiment, a curve of the form $y = f(x)$ is drawn. It is suspected that $f(x)$ is expressible in the form $f(x) = (1+ax)^n$, where a is real and n is a positive integer.

For values of x so small that third (and higher) powers of x may be ignored, the parabola

$$y = 1 - 6x + 16x^2$$

is practically identical with the given curve. Assuming the curves coincide for these values of x, what are the values of a and n?

(ii) It is known that a set of n (≥ 3) points (x, y), each with $x > 0$, satisfies either a relation of the form

$$y = Ax^\alpha \ (A,\ \alpha \text{ real constants})$$

or a relation of the form

$$y = Bc^x \ (B,\ c \text{ real constants},\ c > 0).$$

Assuming ordinary (squared) graph paper and logarithm tables were available to you, explain how you would determine which form of the relation is the correct one.

Question 5.

(i) Define modulus and conjugate of a complex number $z = x + iy$ (x, y real). Prove that $|z|^2 = z\overline{z}$, and that, for any two complex numbers z_1, z_2,

$$(\overline{z_1 z_2}) = (\overline{z_1})(\overline{z_2}).$$

Deduce that

$$|z_1 z_2| = |z_1||z_2|.$$

(ii) Draw neat, labelled sketches (not on graph paper) to indicate each of the subsets of the Argand diagram described below.

(a) $\{z : |z| \geq 1 \text{ and } 0 \leq \arg z \leq \frac{\pi}{3}\}$.

(b) $\{z : z + \overline{z} > 0\}$.

(c) $\{z : |z - 1| < |z + 1|\}$.

(d) $\{z : |z^2 - \overline{(z^2)}| < 4\}$.

Question 6.

(i) (a) Prove that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$$

for every positive integer n. Deduce that $\sum \frac{1}{\sqrt{n}}$ diverges.

(b) Write down a true statement connecting the statements "$u_n \to 0$ as $n \to \infty$" and "$\sum u_n$ converges".

(c) State (without proof) a result on the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \ (p > 0).$$

(ii) Determine whether the series $\sum_{n=2}^{\infty} \frac{n^3 - n^2}{n^4 - 1}$ converges or diverges.

(iii) If $\sum \left(u_n - \frac{1}{n^2}\right)$ converges, prove that $\sum u_n$ converges.

Question 7.

(i) Show that $\frac{d}{dx} \log(x + \sqrt{1 + x^2}\ ) = \frac{1}{\sqrt{1+x^2}}$. Hence, or otherwise, evaluate $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{\sqrt{1+\sin^2 x}}\, dx$.

(ii) (a) Find

$$\int_0^1 \frac{dx}{e^x + 1}.$$

(b) Assuming $e = 2.7183$, evaluate $\int_1^e (\log x)^2\, dx$ to three decimal places.

(iii) A curve is defined by the equation

$$y = \frac{a}{2}\left[e^{x/a} + e^{-x/a}\right] \ (a > 0).$$

Show that $1 + (y')^2 = y^2/a^2$. Show also that the arc length s between points $(0, a)$ and (x, y) of the curve is given by

$$s = \sqrt{y^2 - a^2}.$$

Question 8.

The mean value theorem of the differential calculus states that if the function f is differentiable on the interval $a \leq x \leq b$, then

$$f(b) - f(a) = (b-a)f'(\xi)$$

for a value ξ satisfying $a << b$.

Indicate by means of a clear sketch the geometrical significance of this result.

Using either this theorem or the mean value theorem of the integral calculus, prove that if $0 < x < y$, then

$$\frac{y-x}{y} < \log y - \log x < \frac{y-x}{x}.$$

Deduce that, if $0 < a < c < b$, then

$$\log a < \log c + \frac{a-c}{c},$$
$$\log b < \log c + \frac{b-c}{c}.$$

Prove that if $0 < \lambda < 1$, and if $a < b$, then $a < \lambda a + (1-\lambda)b < b$. Deduce that if $0 < a < b$, and if $0 < \lambda < 1$, then

$$a^{\lambda} b^{1-\lambda} < \lambda a + (1-\lambda)b.$$

Question 9.

An equilateral triangle has vertices $A(0,2), B(-\sqrt{3},-1)$ and $C(\sqrt{3},-1)$, so that its centroid is at the origin. Describe in geometrical terms the six linear transformations of the plane (including the identity transformation) which are symmetry operations for this triangle (i.e., which transform the triangle onto itself).

Write down the 2×2 matrix corresponding to each of these symmetry operations.

Evaluate any four matrix products $\mathbf{AB}$, where $\mathbf{A} \neq \mathbf{1}, \mathbf{B} \neq \mathbf{1}, \mathbf{A} \neq \mathbf{B}$, and $\mathbf{A}, \mathbf{B}$ are chosen from matrices in your list just written down. Verify that the products obtained by you occur in your list, and explain why this is so.

Question 10.

(i) Determine the real values of k for which the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 2k-1 \end{bmatrix}$$

has:

(a) non-zero eigenvalues of the same sign;

(b) non-zero eigenvalues of opposite sign.

(ii) Describe the type of graph represented by the equation $x^2+4xy+(2k-1)y^2=1$ for each of the cases $k=3, k=\frac{5}{2}, k=1$.

(iii) For the case $k=1$, find the principal axes and the foci of the corresponding curve.