NSW HSC 4 Unit Mathematics Examination 1978

1. (i) Use Euclid's algorithm to find the highest common factor of 1247 and 2537 and show that it can be expressed in the form

$$1247p + 2537q$$

in an infinite number of ways (with p, q as integers).

(ii) If a, b, c are given integers prove that the equation

$$ax + by = c$$

has integers solutions for (x, y) if and only if d divides c where d is the highest common factor of a, b.

If (x_0, y_0) is one such solution prove that any other solution must be expressible in the form

$$x = x_0 + \frac{nb}{d},\ y = y_0 - \frac{na}{d}$$

where n is an integer.

2. (i) If α, β, γ are the three roots of

$$x^3 + px + q = 0$$

(p, q being constants) find the cubic equation whose roots are $\alpha^2, \beta^2, \gamma^2$.

(ii) When a polynomial $P(x)$ is divided by $x-2$ and by $x-3$ the respective remainders are 4 and 9; determine what the remainder must be when it is divided by $(x-2)(x-3)$.

3. (i) For a particle of mass m moving in a straight line under the action of a variable force prove that, over a given interval, the work done by the force is equal to the increase in kinetic energy.

(ii) A mass of 20 kilograms hangs from the end of a rope and is hauled up vertically from rest by winding up the rope. The pulling force in the rope starts at 250 newtons and decreases uniformly by 10 newtons for every metre wound up. Find the velocity of the mass after 10 metres have been wound up. (Take the acceleration due to gravity as 9.8 m/s^2 and neglect the weight of the rope.)

4. (i) A, B, C are three points on horizontal ground forming a triangle with a right-angle at C and the lengths of the sides BC, CA are a, b respectively. Vertical borings are made at A, B, C and coal is struck at points P, Q, R at respective depths p, q, r. If θ is the inclination of the plane PQR to the horizontal prove that

$$\tan^2\theta = \frac{(q-r)^2}{a^2} + \frac{(r-p)^2}{b^2}.$$

(ii) The two spheres

$$x^2 + y^2 + z^2 + 2x - 2y = 2$$
$$x^2 + y^2 + z^2 = 1$$

cut in a circle β. Find the equation of the sphere that contains this circle β and also passes through the origin.

5. Define the modulus $|z|$ of a complex number z.

Given two complex numbers z_1, z_2 prove that:

(i) $|z_1 z_2| = |z_1||z_2|$;

(ii) the triangle represented (on the Argand diagram) by the complex numbers $0, 1, z_1$ is similar to the triangle represented by $0, z_2, z_1 z_2$;

(iii) $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2|z_1|^2 + 2|z_2|^2$ and interpret this as a theorem on a triangle giving the length of a median in terms of the lengths of the sides.

6. (i) Prove that the series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is divergent by first proving that the sum of the first 2^n terms exceeds $n/2$.

Also prove that the sum of the first thousand term is less than 10.

(ii) Investigate for convergence the series whose n^{th} term is

(a) $\frac{x^n}{2n+1}$ (where x is positive);

(b) $\sqrt{n^2+1} - n$.

7. Evaluate (giving numerical answers to three significant figures):

(i) $\int_0^1 \frac{dx}{(x+1)(x+3)}$;

(ii) $\int_0^1 \sqrt{4 - x^2}\ dx$;

(iii) $\int_0^1 2^x\ dx$;

(iv) $\int_0^{\frac{1}{2}} \sin^{-1} x\ dx$.

8. In the cartesian plane sketch the curve

$$y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

and prove that the lines $y = \pm 1$ are asymptotes.

Also, if k is a positive constant, find the area in the positive quadrant enclosed by the above curve and the three lines

$$y = 1, \quad x = 0, \quad x = k$$

and prove that this area is always less than $\log_e 2$, however large k may be.

9. In the cartesian plane the point P, whose co-ordinates are (x, y), is transformed to the point P', whose co-ordinates are (x', y'), by the affine transformation $\mathbf{r}' = \mathbf{A}\mathbf{r}$ where $\mathbf{A}$ is a matrix and

$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{r}' = \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

Determine the matrix $\mathbf{A}$ in the following cases:

(i) P' is obtained from P by reflection in the line

$$y = x \tan\theta;$$

(ii) P' is the mid-point of the segment OP (O being the origin);

(iii) P' is the foot of the perpendicular from P on the line

$$ax + by = 0 \quad \text{(where } a, b \text{ are constants)}.$$

10. (i) In the cartesian plane the (x, y) co-ordinate frame is rotated about the origin anti-clockwise through an angle θ to give a new frame with co-ordinates denoted by (x', y'). Derive equations expressing the transformation between the two sets of co-ordinates (x, y) and (x', y').

(ii) Prove that (in cartesian co-ordinates)

$$2x^2 + 4xy - y^2 = 6$$

represents an hyperbola. Illustrate its shape and position by a clear diagram showing, in particular, its asymptotes and principal axes. Also, write down the equations of:

(a) the asymptotes;

(b) the principal axes.