NSW HSC 4 Unit Mathematics Examination 1977

1. (i) If a, b, c are given integers prove that the equation

$$ax + by = c$$

has solutions in integers if and only if c is divisible by the highest common factor of a and b.

(ii) Show that the equation

$$357x + 323y = 408$$

has integer solutions for x, y and find one such solution.

2. (i) Given that the three roots of the equation

$$8x^2 - 36x^2 + 38x - 3 = 0$$

are in arithmetic progression, find them.

(ii) Decompose

$$\frac{1}{(x+1)(x^2+4)}$$

into partial fractions and use this result to evaluate

$$\int_0^2 \frac{1}{(x+2)(x^2+4)}\ dx.$$

3. Prove that a particle projected with velocity V at an angle α with the horizontal describes a parabola, and find an expression for the focal length in terms of V, α, g where g is the acceleration due to gravity (assumed constant).

Several particles are projected from the same point simultaneously with the same initial speed U in various directions. Prove that at any subsequent time t they all lie on a sphere of radius Ut.

4. The function f is defined over the domain of real numbers by the relations

$$f(x) = x \text{ when } 0 \leqslant x \leqslant 1$$

and

$$f(x) = 0 \text{ for all other values of } x.$$

If k is a given constant evaluate

(i) $\int_0^k f(x)\ dx$;

(ii) $\int_{-1}^1 f(x)f(x-k)\ dx$.

5. (i) Given

$$w = \frac{2-3i}{1+i}$$

determine

(a) $|w|$ (i.e., the modulus of w);

(b) $\overline{w}$ (i.e., the conjugate of w);

(c) $w + \overline{w}$.

(ii) Describe, in geometric terms, the locus (in the Argand plane) represented by $2|z| = z + \overline{z} + 4$.

6. (i) Explain the terms "convergent" and "absolutely convergent" as applied to an infinite series of real numbers.

(ii) Prove, quoting carefully any theorems or tests used, that:

(a) for $-1 < r < 1$ any α arbitrary, the series

$$1 + r\cos\alpha + r^2\cos 2\alpha + \cdots + r^n \cos n\alpha + \cdots$$

is convergent;

(b) the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots$$

is convergent, but not absolutely convergent. Also show that its sum lies between 5/6 and 7/12.

7. (i) Define $\sin^{-1} x$ and show that its derivative is

$$\frac{1}{\sqrt{1-x^2}}.$$

(ii) Evaluate (giving numerical answers to three significant figures):

(a) $\int_0^1 \frac{dx}{\sqrt{9-4x^2}}$;

(b) $\int_0^\pi \sin^2(x/4)\, dx$;

(c) $\int_0^1 x^2 e^{-x}\, dx$.

8. Sketch (in the Cartesian x, y plane) the curve

$$3cy^2 = x(x-c)^2$$

where c is a positive constant.

Denoting that part of this curve which lies in the positive quadrant between $x = 0$ and $x = c$ by β prove that:

(i) the area enclosed between β and the x-axis is

$$\frac{4\sqrt{3}c^2}{45};$$

(ii) the length of β is $2c/\sqrt{3}$.

9. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}.$$

(i) Write down the characteristic equation of $\mathbf{A}$.

(ii) Prove that $\mathbf{A}^2 - 10\mathbf{A} + 9\mathbf{I} = \mathbf{0}$.

(iii) Find $\mathbf{A}^{-1}$.

(iv) Find the eigenvalues of $\mathbf{A}$.

(v) For the eigenvalues of $\mathbf{A}$ find the corresponding eigenvectors.

(vi) Deduce (without resorting to calculations similar to those done in the preceeding parts) the eigenvalues and eigenvectors of $\mathbf{A}^2$.

10. (i) Explain, with proof and the aid of a diagram, why

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}$$

is called a reflection matrix.

(ii) The Cartesian equation of a given conic x is

$$ax^2 + 2hxy + by^2 = 1;$$

show that this equation can be written in the matrix form

$$\mathbf{r}^{\mathrm{T}}\mathbf{A}\mathbf{r} = \mathbf{1} \text{ where } \mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}$$

and specify the matrix $\mathbf{A}$ explicitly in terms of a, h, b.

(iii) The reflection of the above conic in the line $y = mx$ is another conic s'; prove that the equation of s' can be expressed as

$$\mathbf{r}^{\mathrm{T}}\mathbf{S}\mathbf{A}\mathbf{S}\mathbf{r} = \mathbf{1}$$

and find the matrix $\mathbf{S}$ explicitly in terms of m.