NSW HSC 4 Unit Mathematics Examination 1976.

1. (i) Use Euclid's algorithm to find the highest common factor (i.e., the greatest common divisor) of 7313 and 6319 and express the result in the form

$$7313p + 6319q$$

where p and q are integers.

(ii) a, b, c are three given integers. If d is the highest common factor of a, b and D is the highest common factor of c, d prove that D is the highest common factor of the three integers a, b, c.

2. (i) It is given that the constants p, q are such that the equation

$$x^3 + px + q = 0$$

has three real non-zero roots α, β, γ. Evaluate (in terms of p, q)

(a) $\alpha^2 + \beta^2 + \gamma^2$;

(b) $\alpha^4 + \beta^4 + \gamma^4$;

(c) $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$;

and prove that p must be strictly negative.

(ii) Use Euclid's algorithm to show that the polynomials (over the field of real numbers)

$$1 + x + x^2 \text{ and } 1 + x^2$$

are relatively prime and hence obtain a decomposition of

$$\frac{1}{(1 + x + x^2)(1 + x^2)}.$$

into the sum of two partial fractions.

3. A particle of unit mass is projected vertically upwards from the ground with initial speed u. If the air resistance at any instant is proportional to the velocity at that instant (say, resistance $= -kv$) prove that the particle reaches its highest point in time T given by

$$kT = \log(1 + ku/g)$$

where g is the acceleration due to gravity (assumed constant).

If the highest point reached is at a height h above the ground prove that

$$hk = u - gT.$$

4. (i) Using graphical considerations, or otherwise, prove that, for $x > 0$,

$$\log_e x \leqslant x - 1.$$

(ii) If $\{p_1, p_2, p_3, \ldots, p_n\}$ is a set of n positive numbers adding to unity, i.e.,

$$p_1 + p_2 + p_3 + \cdots + p_n = 1 \text{ and each } p_r > 0,$$

prove that

$$\sum_{r=1}^{n} \log(np_r) \leqslant 0.$$

(iii) Given n positive numbers $x_1, x_2, x_3, \ldots, x_n$ prove that

$$x_1 x_2 x_3 \ldots x_n \leqslant \left[\tfrac{1}{n}(x_1 + x_2 + x_3 + \cdots + x_n)\right]^n.$$

Also investigate the case of equality in this relation.

5. (i) If z is the complex number $x + iy$ (where x and y are real numbers) define

(a) $|z|$ (the modulus of z);

(b) $\overline{z}$ (the conjugate of z);

(c) $\Re(z)$ (the real part of z).

Also prove that, for any two complex numbers z, w,

$$|z + w| \leqslant |z| + |w|.$$

(ii) Given

$$w = \frac{1 + 2i}{3 + 4i}$$

determine

(a) $|w|$;

(b) $\overline{w}$;

(c) $\Re(w)$;

(d) $\Re(2w + \overline{w})$.

(iii) Describe, in geometric terms, the loci (in the Argand plane) represented by

(a) $|z - 2i| = 1$;

(b) $z\overline{z} = z + \overline{z}$.

6. Prove that the series

$$1 - \frac{x^2}{(1!)^2} + \frac{x^4}{(2!)^2} - \frac{x^6}{(3!)^2} + \cdots + \frac{(-1)^n x^{2n}}{(n!)^2} + \cdots$$

is convergent for all values of x. (Any theorem or test used should be quoted carefully.)

Denoting the sum of this series by $f(x)$ prove that $f(1) > 0$ and $f(2) < 0$.

It appears that the equation $f(x) = 0$ has a root lying between 1 and 2; describe how you would make a guess at the value of this root.

7. (i) Describe briefly the content and significance of the Fundamental Theorem of the Calculus.

(ii) Evaluate

(a) $\int_0^1 xe^{-x}\ dx$;

(b) $\int_1^2 x \log_e x\ dx$;

(c) $\int_0^2 \frac{1+x}{4+x^2}\ dx$;

(d) $\int_0^{\frac{1}{4}} \frac{dx}{\sqrt{1-4x^2}}$.

8. In the Cartesian plane, indicate, by shading, the region R whose points simultaneously satisfy the relations

$$0 \leqslant x \leqslant 2, \quad 0 \leqslant y \leqslant 4x^2 - x^4;$$

and determine the maximum value of y ffor"333UGbv,"9CUyboin,9CUCyts oi

Also prove that the rotation $\mathcal{T}$ above can be factorised into a product of two reflections in an infinite number of ways.

10. In the Cartesian plane prove that

$$x^2 + xy + y^2 = 6$$

represents an ellipse and find

(i) the length of its major axis (i.e., the longer axis);

(ii) the equation of the line in which the major axis lies;

(iii) the equations of its two tangents which are parallel to the x-axis.

Illustrate by a sketch.