NSW HSC 4 Unit Mathematics Examination 1975

Question 1.

Sketch the graphs (showing the main features - do NOT use squared paper) of:

(i) $y = \cos^2(x/2) \quad (0 \leqslant x \leqslant 4\pi)$;

(ii) $y = e^{-x^2}$;

(iii) $y = \log|x|$;

(iv) $y = \frac{\sin x}{x}$;

(v) $y = \max(x, 1-x)$ wher $\max(a, b)$ denote the greater of the two numbers a and b, i.e.,

$$\begin{aligned} \max(a,b) &= a \text{ if } a \geqslant b \\ &= b \text{ if } a < b. \end{aligned}$$

Question 2.

(i) Define "the greatest common divisor" of two integers a, b.

(ii) Use Euclid's algorithm to compute the greatest common divisor of 1081 and 943.

(iii) Given two positive integers a, b with their greatest common divisor d expressed in the form

$$d = pa + qb$$

where p and q are integers, determine what value or values the greatest common divisor of p and q can have.

Question 3.

(i) Evaluate:

(a) $\int_0^1 te^{-2t}\, dt$; (b) $\int_{-1}^1 \frac{dx}{(x+2)(x+5)}$;

(c) $\int_0^1 x^2\sqrt{1-x^2}\, dx$.

(ii) Evaluate:

$$\lim_{h\to 0} \frac{2^h - 1}{h}$$

Question 4.

In the Cartesian plane, R denotes the region consisting of those points whose co-ordinates (x, y) simultaneously satisfy the relations

$$y \leqslant \sin x, \quad 0 \leqslant x \leqslant \pi, \quad y \geqslant 0.$$

Prove that the area of R is 2(units of area) and calculate the volume of the solid formed when R is rotated (through one complete revolution) about

(i) the x-axis; (ii) the y-axis.

Question 5.

(i) A particle moves in the x-axis with acceleration at position x given by

$$\frac{d^2x}{dt^2} = -p^2x$$

where p is a positive constant. If at zero time the particle is released from rest at the point $x = a$ derive formulae for

(a) its velocity v at position x;

(b) its position x at time t.

(ii) The rise and fall of the tide at a certain harbour may be taken as simple harmonic, the interval between successive high tides being 12.5 hours. The harbour entrance has a depth of 15 metres at high tide and 7 metres at low tide. If low tide occurs at 11 a.m. on a certain day find the earliest time thereafter that a ship requiring a minimum depth of 13 metres of water can pass through the entrance.

Question 6.

(i) Explain, with proof and with the aid of a diagram, why

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}$$

is called a reflection matrix.

(ii) Prove that the product of two reflection matrices represents a rotation.

(iii) Given $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ compute the elements of the matrix A^{8n+4} where n is an integer.

Question 7.

(i) For the matrix $\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ find the eigenvalues and corresponding eigenvectors.

(ii) Prove that, in the Cartesian plane, the equation

$$5x^2 + 6xy + 5y^2 + 4x - 4y = 0$$

represents an ellipse and determine

(a) the co-ordinate of its centre;

(b) the equations of its principal axes.

Illustrate by a sketch.

Question 8.

(i) For the sphere (in three-dimensional Cartesian space)

$$x^2 + y^2 + z^2 - 2x - 6y + 2z + 2 = 0$$

find

(a) the co-ordinates of the centre;

(b) the equation of the tangent plane at the point $(3, 4, 1)$.

(ii) S is a given sphere and β a given plane (not intersecting S). With any point P on β as centre, a sphere is drawn with radius equal to the length of the tangent from P to S. Prove that all such spheres have two points in common.

Question 9.

(i) Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ is convergent for $s > 1$.

(ii) Determine whether the series

$$\sum_{n=1}^{\infty} \frac{2n+3}{n^3+1}$$

is convergent or divergent.

(iii) It follows from part (i) that the series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots$$

is convergent and, in fact, it is known that its sum is $\pi^2/6$. It is also given that the sum of the first thousand terms of this series is 1,644 (with four-figure accuracy). Hence we obtain an approximation to the value of π from the following calculations:

$$6(1.644) = 9.864$$

$$\sqrt{9.864} = 3.141 \text{ (to four-figure accuracy)}$$

and therefore $\pi = 3.141$ approximately.

Without using any other knowledge of the value of π, investigate the accuracy of this approximation so obtained for π.

Question 10.

(i) If w is a complex number $i - 1$ indicate on the Argand diagram the points

$$w, \overline{w}, iw, \tfrac{1}{w}.$$

(ii) On the Argand diagram, P represents the complex number z and R the number $\frac{1}{z}$; and a square $PQRS$ is drawn in the plane with PR as a diagonal. If P lies on the circle $|z| = 2$ prove that Q will lie on an ellipse whose equation has the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and specify numerical values for a and b.