NSW HSC Mathematics Level 1 (4 Unit) Examination 1974

Question 1.

(i) Sketch the graphs (showing the main features - do NOT use squared paper) of:

(a) $y = |\sin x|$ $(-2\pi \leqslant x \leqslant 2\pi)$;

(b) $y = \sin|x|$ $(-2\pi \leqslant x \leqslant 2\pi)$;

(c) $(x+2)(y+1) = 1$.

(ii) Assuming that $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$
evaluate

(a) $\lim\limits_{x\to 0} \left[\frac{\sin^2(3x)}{x^2}\right]$;

(b) $\lim\limits_{x\to 0} \left[\frac{1-\cos(4x)}{x^2}\right]$.

Question 2.

State the Fundamental Theorem of Arithmetic (concerning the factorisation of integers into primes) and give a proof of this theorem by first showing that, if a prime p divides the product ab of two integers and does not divide a, then it must divide b.

Question 3.

(i) If z is the complex number $1 + 2i$ indicate on the Argand diagram the points

$$z, \overline{z}, z^2, \frac{1}{z}.$$

(ii) On the Argand diagram P represents the complex number z and Q the complex number $\frac{1}{z}$. If P lies on the straight line $x = 1$ prove that Q will lie on a certain circle and find its centre and radius.

Question 4.

In the Cartesian plane indicate (by shading) the region R consisting of those points whose coordinates (x, y) simultaneously satisfy the five relations

$$0 \leqslant x \leqslant \pi/2,\ y \geqslant 0,\ y \geqslant \sin x,\ y \leqslant \cos x,\ y \leqslant \tan x.$$

Also prove that the area of R is

$$\tfrac{1}{2}\left[2\sqrt{2} - 1 - \sqrt{5} - \log_e\left\{\tfrac{\sqrt{5}-1}{2}\right\}\right].$$

Question 5.

(i) Find the length of the arc of the curve

$$y = \frac{x^4 + 3}{6x}$$

that lies between $x = 1$ and $x = 2$.

(ii) If

$$f(x) = \int_0^{2x} \sqrt{1 + t^4}\, dt$$

evaluate $f'(2)$ where $f'(x)$ denotes the derivative of $f(x)$.

Question 6.

(i) State the values of x for which the series

$$1 - x^2 + x^4 - \cdots + (-1)^{n-1} x^{2n-2} + \cdots$$

is convergent.

Also write down an expression for the sum of its first n terms.

(ii) Determine all the values of x for which the series

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + \cdots$$

is convergent. Also prove that, when it is convergent, its sum is $\tan^{-1} x$.

Question 7.

In three-dimensional Cartesian space, S denotes the sphere

$$x^2 + y^2 + z^2 = 1$$

and β the plane

$$2x + 2yz = 1.$$

(i) Find the equations of the two planes which are parallel to β and which touch S (i.e., are tangent planes to S).

(ii) If β meets the plane $z = 2$ in the line p find the equations of the two tangent planes to S which pass through p.

Question 8.

Given that

$$\mathbf{A} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix},$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \mathbf{d} = \begin{bmatrix} 3 \\ 0 \end{bmatrix},\ \mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix},\ \mathbf{r}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

describe in geometrical language (using terms such as reflection, rotation, etc.) the transformation $\mathbf{r} \to \mathbf{r}'$ in the Cartesian plane in the following cases:

(i) $\mathbf{r}' = \mathbf{A}\mathbf{r}$;

(ii) $\mathbf{r}' = \mathbf{B}\mathbf{r}$;

(iii) $\mathbf{r}' = \mathbf{B}\mathbf{r} + \mathbf{d}$;

(iv) $\mathbf{r}' = \mathbf{C}\mathbf{r}$;

(v) $\mathbf{r}' = (\mathbf{I} + \mathbf{C})^2\mathbf{r}$.

Question 9.

The matrix $\mathbf{A}$ is given by $\mathbf{A} = \frac{1}{2}\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$.

(i) Show that the eigenvalues of $\mathbf{A}$ are 4,1 and find the corresponding eigenvectors.

(ii) Find a matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix.

(iii) Solve, for $\mathbf{X}$, the matrix equation $\mathbf{X}^2 = \mathbf{A}$.

Question 10.

A large vertical wall stand on horizontal ground. The nozzle of a water hose is positioned at a point C on the ground at a distance c from the wall and the water jet can be pointed in any direction from C. Also the water issues from the nozzle with speed V. (Air resistance may be neglected and the constant g denotes the acceleration due to gravity.)

(i) Prove that the jet can reach the wall above ground level if and only if $V > \sqrt{gc}$.

(ii) If $V = 2\sqrt{gc}$ prove that the portion of the wall that can be reached by the jet is a parabolic segment of height $15c/8$ and area $5\sqrt{15}c^2/2$.