NSW HSC Mathematics Level 1 (4 Unit) Examination 1973

Question 1.

(i) Sketch the graphs (showing the main features - do NOT use squared paper) of:

(a) $y = \sin^2(2x)$ $(-2\pi \leqslant x \leqslant 2\pi)$;

(b) $\sin(x + y) = 0$;

(c) $|x| + |y| = 1$.

(ii) Evaluate $\int_0^1 x \tan^{-1} x \; dx$.

Question 2.

(i) Define "the greatest common divisor of two integers a, b".

(ii) Given two odd integers a, b satisfying a relation

$$pa + qb = 8$$

where p, q are integers prove that a and b are co-prime (i.e., their greatest common divisor is 1).

(iii) Prove that two integers m, n must be co-prime if $m+n$ and $m-n$ are co-prime.

Question 3.

(i) Define the modulus $|z|$ of a complex number z.

(ii) If z_1 and z_2 are two complex numbers prove that

$$|z_1 z_2| = |z_1||z_2|.$$

(iii) In the Argand diagram P represents the complex number z and Q the complex number ζ given by:

$$\zeta = \frac{3z - 1}{z - 1}.$$

If P describes the circle of unit radius with centre at origin find the locus described by Q.

Question 4.

Prove that the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \; (a \geqslant b > 0)$$

is πab.

Also find the volume of the solid formed when this area is rotated through one complete revolution about the line $y = b$.

Question 5.

Explain and derive Simpson's Rule (for three ordinates), namely,

$$\int_a^b f(x)\ dx = \frac{b-a}{6}[f(a) + 4f(\tfrac{1}{2}(a+b)) + f(b)] \text{ approximately.}$$

Also prove that Simpson's Rule yields an exact result when $f(x)$ is a cubic polynimial, i.e., when it has the form

$$f(x) = Ax^3 + Bx^2 + Cx + D.$$

Question 6.

(i) Show that, in the Cartesian plane, a change in coordinates from (x, y) to (x^*, y^*) produced by rotating the axes through an angle α can be expressed by the transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x^* \\ y^* \end{bmatrix}.$$

(ii) Assume that, by a suitable transformation of the above type, the equation

$$ax^2 + 2hxy + by^2 = 1$$

becomes

$$\lambda_1 {x^*}^2 + \lambda_2 {y^*}^2 = 1$$

where λ_1, λ_2 are the eihenvalues of the matrix

$$\begin{bmatrix} a & h \\ h & b \end{bmatrix}.$$

State and prove *necessary and sufficient* conditions on a, h, b in order that

$$ax^2 + 2hxy + by^2 = 1$$

should represent an ellipse, and, when these conditions are satisfied, prove that the area of the ellipse is $\pi/\sqrt{ab - h^2}$.

(NOTE: Results stated in any previous question on this paper may be assumed.)

Question 7.

(i) With rectangular coordinates in three-dimensional Cartesian space the equations of two planes λ, μ are respectively

$$2x + 3y + 6z = 1$$
$$3x + 2y - 2z = 1.$$

Find:

(a) the angle between the two planes;

(b) the equation of the sphere which touches the plane λ and has its centre at the point $(3, 1, 1)$.

(ii) In the following result (which is valid for $-1 < x < 1$)

$$\log_e \left(\frac{1+x}{1-x}\right) = 2\left[\frac{x}{1} + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots\right]:$$

(a) use the first two terms of the series on the right-hand side to obtain $0{\cdot}69$ as an approximate value of $\log_e 2$;

(b) use the remaining terms of the series to prove that the error in this approximation for $\log_e 2$ is less than 1%.

Question 8.

With the earth regarded as a uniform solid sphere of radius R it is given that the acceleration due to gravity at a point distant r from the earth's centre is directed towards the centre and has magnitude proportional to r^{-2} when $r \geqslant R$ and proportional to r when $r \leqslant R$; and the magnitude of this acceleration at the surface of the earth (i.e., when $r = R$) is denoted by the constant g. Suppose a narrow tunnel is bored along a diameter AB of the earth and a particle is projected from A with initialo velocity U towards B.

(a) Show that the subsequent motion is oscillatory if and only if

$$U^2 < 2gR.$$

(b) When the motion is oscillatory prove that it takes place between two point whose distance apart is

$$\frac{2R}{1 - U^2/(2gR)}.$$

(c) If $U = 0$ state, with brief justification, the period of the motion.

Question 9.

The matrix $\mathbf{C}$ is given by

$$\mathbf{C} = \begin{bmatrix} a & -b \\ -a & b \end{bmatrix}.$$

(i) Find $\mathbf{C}\mathbf{C}^{\mathrm{T}}, \mathbf{C}^{\mathrm{T}}\mathbf{C}$ and $\mathbf{C}^2$.

(ii) If n is a positive integer $\geqslant 2$ and $\lambda = a + b \neq 0$ prove that

(a) $\mathbf{C}^n = \lambda^{n-1}\mathbf{C}$;

(b) $(\mathbf{1} - \mathbf{C})^n = \mathbf{1} - \frac{1}{\lambda}\{1 - (1 - \lambda)^n\}\mathbf{C}$.

(NOTE: $\mathbf{C}^{\mathrm{T}}$ denotes the transpose of $\mathbf{C}$ and $\mathbf{1}$ the unit matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.)

Question 10.

A man inherits, on his 21st birthday (31 December, 1970), two houses - one in the city and one in the country. On the first day after this birthday, i.e., on 1 January, 1971, he moves into the city house. In this house there is a box containing one red and two white balls and in the country house there is a similar box containing one red and three white balls. Each day he draws at random a ball from the box in the house where he is, notes its colour, and then returns it to the box: if it is red he moves to spend the next day at the other house (otherwise he stays where he is and awaits the outcome of the next drawing).

(i) Calculate the probability that he is in residence at the city house:

(a) on 3 January, 1971;

(b) on the n-th day after his 21st birthday.

(ii) Assuming that he lives to a ripe old age estimate (approximately) the probability that he will die in the city house.

(NOTE: Results stated in any previous question on this paper may be assumed.)