NSW HSC Mathematics Level 1 (4 Unit) Examination 1972

1. (i) Find the greatest common divisor of 713 and 943.

(ii) If a, b, c are given integers prove that the equation

$$ax + by = c$$

has solutions in integers for x, y if and only if c is divisible by the greatest common divisor of a and b.

(iii) Determine whether the equation

$$713x + 943y = 115$$

has integer solutions for x, y and, if it has, find one such solution.

2. Explain how complex numbers are represented on the Argand diagram.

Given that, in the Argand diagram, the point P represents the complex number z and Q the number z^2 prove that if P moves on a straight line parallel to (but not coinciding with) the imaginary axis then Q will move on a certain parabola, and that all such parabolas have a common focus.

Also state what the locus of Q is when P describes the imaginary axis.

3. (i) Evaluate

(a) $\int_0^{\log 2} te^{-t}\ dt$;

(b) $\int_0^{2/3} \frac{dx}{4+9x^2}$.

(ii) The domain of the function f is the interval $-3 \leqslant x \leqslant 7$ and

$$f(x) = (x^3 - 6x^2)^{\frac{1}{2}}$$

Determine the range of the function f and sketch its graph.

Do NOT use squared paper for the graph and give answers for the range to two significant digits.

4. (i) Explain, with proof and with the aid of a diagram, why

$$\begin{bmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{bmatrix}$$

is called a reflection matrix.

(ii) Given that

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where a, b, c, d are real numbers such that

$$\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{1} \text{ and } \det \mathbf{A} = -\mathbf{1}$$

prove that $\mathbf{A}$ is a reflection matrix.

(NOTE.- $\mathbf{A}^{\mathrm{T}}$ denotes the transpose of $\mathbf{A}$ and $\mathbf{1}$ denotes $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.)

5. (i) Describe in geometrical terms the foci (in three-dimensional cartesian space) represented by

(a) $x^2 + y^2 + z^2 + x + y + z = 0$;

(b) $x^2 + y^2 + x + y = 0$.

(ii) Explain the term "direction cosines".

The inclinations to the horiontal of two lines which are perpendicular to one another are α and β. If these two lines lie in a plane which is inclined at an angle θ to the horizontal prove that

$$\sin^2 \theta = \sin^2 \alpha + \sin^2 \beta.$$

6. Explain the terms "convergent sequence" and "convergent series".

Prove that, for $t \neq 1$, the sum of the geometric series

$$1 + t + t^2 + \cdots + t^{n-1}$$

is $(1 - t^n)/(1 - t)$.

By integrating this result between 0 and x prove that, for $0 \leqslant x < 1$, the series

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} + \cdots$$

is convergent and that its sum is $-\log(1 - x)$.

7. The co-ordinates of a point moving in the cartesian (x, y) plane at time t are given by

$$x = t + \sin t, y = 1 - \cos t, \text{ where } 0 \leqslant t \leqslant 2\pi.$$

(i) Show that the curve thus traced out is symmetric about the line parallel to the y-axis through the point where $t = \pi$.

(ii) Determine whether the curve has a tangent at the point where $t = \pi$.

(iii) Find the length of the curve (between $t = 0$ and $t = 2\pi$).

8. (i) With $\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\mathbf{r}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$

let $\mathbf{r}' = \mathbf{Mr}$ be the equation of a affine transformation $\mathcal{T}$ of a cartesian plane into itself in which the points P, Q are transformed into P', Q' respectively. Prove that

(a) $\mathcal{T}$ transforms points of the line segment PQ into points of the line segment $P'Q'$;

(b) area $\triangle OP'Q' = (\text{area } \triangle OPQ).\det \mathbf{M}$.

(ii) Specify an affine transformation which will transform the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b > 0)$$

into a circle and hence, using the results in (i), prove that the area of the ellipse is πab.

9. In this question assume that the earth is a square of radius R and that, at a point distant r $(> R)$ from the centre of the earth, the acceleration due to gravity is proportional to r^{-2} and is directed towards the earth's centre; also, neglect forces due to all causes other than the earth's gravity.

A body is projected vertically upwards from the surface of the earth with initial speed V.

(i) Prove that it will escape from the earth (i.e., never return) if and only if $V \geqslant \sqrt{2gR}$ where g is the magnitude of the acceleration due to gravity at the earth's surface.

(ii) If $V = \sqrt{2gR}$ prove that the time taken to rise to a height R above the earth's surface is $\frac{1}{3}(4 - \sqrt{2})\sqrt{R/g}$.

10. (i) Explain what is meant by a "root of multiplicity m of a polynomial".

(ii) If a polynomial $f(x)$ has a root of multiplicity m at $x = c$ prove that the derived polynomial $f'(x)$ has a root of multiplicity $(m-1)$ at the same point.

(iii) Prove that the polynomial

$$x^3 + 3px^2 + 3qx + r,$$

has a double root (i.e., a root of multiplicity 2) if and only if

$$(pq - r)^2 = 4(p^2 - q)(q^2 - pr) \text{ and } p^2 \neq q.$$