NSW HSC Mathematics Level 1 (4 Unit) Examination 1971

1. (i) Define $\sin^{-1} x$ and show that its derivative is $(1-x^2)^{-\frac{1}{2}}$.

(ii) Evaluate:

(a) $\int_1^2 \frac{dx}{x(x+3)}$;

(b) $\int_0^2 \frac{(x+1)\ dx}{x^2+4}$;

(c) $\int_0^1 \sin^{-1} t\ dt$.

2. A clock whose face is in the x, y-plane is movede bodily in the x-direction with a constant speed of 1 foot per second. If the clock has a sweep second hand of length 1 foot (making one complete revolution per minute) prove that the tip of this hand describes a curve whose tangent varies in inclination to the x-axis between $+\alpha$ and $-\alpha$ where

$$\tan\alpha = \left(\frac{900}{\pi^2 - 1}\right)^{-\frac{1}{2}}.$$

3. (i) Prove that the area of the parabolic segment enclosed between $y = x^2$ and the line $y = x$ is $1/6$.

(ii) Find the volume of the solid formed when the area in (i) is rotated (through one complete revolution) about the x-axis.

(iii) Write down the perpendicular distance from the point (t, t^2) to the line $y = x$.

(iv) Find the volume of the solid formed when the area in (i) is rotated (through one complete revolution) about the line $y = x$.

4. (i) Use Euclid's algorithm to find the greatest common divisor d of 221 and 104.

Also show that d can be expressed in the form

$$d = 221m + 104n$$

where m, n are integers and, moreover, that this can be done in an infinite number of ways.

(ii) What can you conclude about the greatest common divisor of two integers a, b if integers p, q exist such that

$$ap + bq = 6?$$

5. Which of the following statements (on infinite series of real numbers) are true and which false? Justify your answers by proving those that are true and by giving a counterexample for any one that is false.

(i) If $\sum u_n$ and $\sum v_n$ are convergent then so is $\sum(u_n + v_n)$.

(ii) If $\sum u_n$ and $\sum v_n$ are divergent then so is $\sum(u_n + v_n)$.

(iii) If $\sum u_n$ is convergent and $0 < u_n < 1$ (for all n) then

$$\sum \frac{u_n}{1 - u_n} \text{ is convergent.}$$

(iv) If $\sum u_n$ is convergent then so is $\sum u_n^2$.

6. The plane $6x + 2y + 3z = 42$ cuts the x, y, z axes respectively in A, B, C. Find:

(i) the equation of the sphere whose centre is O (the origin) and which touches the given plane;

(ii) the equation of the sphere which passes through the four points O, A, B, C;

(iii) the equation of the cylinder which passes through the curve of intersections of the above two spheres and whose generators are parallel to the x-axis.

7. The acceleration of a body moving along the x-axis is given by:

$$\begin{aligned} \frac{d^2x}{dt^2} &= -x \qquad \text{for } x \leqslant 2; \\ &= 4 - x \quad \text{for } x > 2. \end{aligned}$$

(i) If U is the function given by

$$U(x) = \frac{1}{2}x^2 \text{ for } x \leqslant 2 \text{ and } U(x) = \frac{1}{2}(x-4)^2 \text{ for } x > 2$$

deduce the fact that the quantity

$$E = \frac{1}{2}(dx/dt)^2 + U(x)$$

is a constant of the motion (i.e., is independent of the time).

(ii) At time $t = 0$ the body is placed at the origin and is given an initial velocity $v_0 = \frac{1}{2}$ in the positive x-direction. Determine the extreme points (largest and smallest x values) of the subsequent motion; hence show that the motion is simple harmonic. (Hint: Use the result in (i).)

(iii) With the same starting position, but a larger initial velocity $v_0 = 3$, find the extreme points of the subsequent motion. Is this motion simple harmonic? Explain your answer fully.

8. (i) Show that, in the Cartesian plane, a change in coordinates from (x, y) to (x^*, y^*) produced by rotating the axes through an angle α can be expressed by the transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x^* \\ y^* \end{bmatrix}.$$

(ii) Prove that, in the Cartesian plane, the equation

$$x^2 - 4xy - 2y^2 = 1$$

represents an hyperbola and find the equations of its principal axes. Illustrate by a sketch.

9. (i) Explain with the aid of a diagram (without proof) how the matrix

$$\begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}$$

represents a reflection.

(ii) In the Cartesian plane the reflection of the point P in the line

$$3x - 4y = 10$$

is the point P' If the coordinates of P and P' are (x, y) and (x', y') respectively and if

$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix} \ , \ \mathbf{r}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

derive a matrix expressing $\mathbf{r}'$ in terms of $\mathbf{r}$.

10. (i) Given a complex number $z = x + iy$ (where x and y are real) define the modulus $|z|$ and the conjugate $\overline{z}$.

(ii) Specify the geometric locus in the complex (Argand) plane represented by the equation

$$|z - z_0| = c$$

where z_0 is a fixed complex number and c a real constant.

(iii) Given that, in the complex plane, the point P represents the complex number z and Q the number $1/z$ prove that, if P describes a circle of radius r with centre at w (where $r \neq |w|$), then Q will also describe a circle, whose centre is at the point

$$\frac{\overline{w}}{|w|^2 - r^2};$$

and find the radius of this circle.