NSW HSC Mathematics Level 1 (4 Unit) Examination 1970

1. Evaluate:

(i) $\int_0^1 \frac{du}{\sqrt{4-u^2}}$;

(ii) $\int_0^1 \frac{dx}{4-x^2}$;

(iii) $\int_0^{\pi/4} \theta \cos^2 \theta \ d\theta$;

(iv) $\int_0^1 te^{-t^2} \ dt$.

2. The minute hand OP and hour clock OQ of a clock are 4 feet and 3 feet long respectively. At the instant when the clack shows 9 o'clock find the rate (in feet per hour) at which the length PQ is increasing.

3. (i) Calculate the length of the arc of the curve $y = 2x\sqrt{x}$ between $x = 0$ and $x = 1$.

(ii) Prove that the volume common to the solid cylinders

$$x^2 + z^2 \leqslant 1 \text{ and } y^2 z^2 \leqslant 1$$

(where x, y, z are rectangular cartesian coordinates) is 16/3.

[Hint: Consider sections by planes parallel to the xy-plane].

4. If a particle moving with speed v experiences air resistance kv^2 per unit (k being a constant) prove that, in falling from rest in a vertical line through a distance s, it will acquire a speed

$$V\sqrt{1 - e^{-2ks}}$$

where $V = \sqrt{g/k}$ (the "terminal velocity") and g is the acceleration due to gravity (assumed constant).

With air resistance as above prove that a particle projected vertically upwards with initial speed U will return to the point of projection with speed W given by

$$W^{-2} = U^{-2} + V^{-2}.$$

5. Explain the terms "convergent" and "absolutely convergent" as applied to an infinite series of real numbers and prove that an absolutely convergent series is convergent.

Determine for what (real) values of x the series

$$\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} + \cdots$$

is:

(i) convergent

(ii) absolutely convergent.

6. The plane $3x + 2y + 6z = 12$ cuts the x, y, z axes respectively in A, B, C. Find the equation of the sphere which passes through the four points O, A, B, C (where O is the origin) and specify its centre and radius.

Also find the radius of the circumcircle of the triangle ABC.

7. (i) Find the eigenvalues and corresponding eigenvectors of the matrix

$$\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}.$$

(ii) f is a function whose domain is the set of points in the cartesian (x, y) plane excluding the origin and

$$f(P) = \frac{3x^2 - 2xy + 3y^2}{x^2 + y^2}$$

where P is the point (x, y).

Find the range of the function f.

8. Show that, in a cartesian (x, y) frame, a rotation $\mathcal{T}$ through an angle α about the origin (thereby transforming $\mathbf{r}$ into $\mathbf{r}'$) can be represented by the matrix equation $\mathbf{r}' = \mathbf{Tr}$ where

$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad \mathbf{T} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

If $\mathcal{R}$ denotes a refection in the line $y = x\tan\beta$ (thereby transforming $\mathbf{r}$ into $\mathbf{r}'$) find the matrix $\mathbf{R}$ such that $\mathbf{r}' = \mathbf{Rr}$.

Also form the matrix products $\mathbf{RT}$ and $\mathbf{TR}$ and hence describe geometrically the transformation $\mathcal{RT}$ and $\mathcal{TR}$.

9. (i) Define the modulus $|z|$ of aq complex number z and show that every non-zero complex number z can be written in the form $r(\cos\theta + i\sin\theta)$ where $r = |z|$.

(ii) Express $\frac{1}{1+i}$ in the form described in (i).

(iii) The four complex numbers z_1, z_2, z_3, z_4 are represented on the complex (Argand) plane by the points A, B, C, D respectively. If $z_1 - z_2 + z_3 - z_4 = 0$ and $z_1 - iz_2 - z_3 + iz_4 = 0$, determine the possible shape(s) for the quadrilateral $ABCD$.

10. Given that $\alpha, \beta, \gamma, \delta$ are the roots of the equation

$$\frac{x}{x-p} + \frac{x}{x-q} + \frac{x}{x-r} + (x - s) = 0,$$

where p, q, r, s are distinct non-zero constants:

(i) evaluate $\alpha + \beta + \gamma + \delta$ in terms of p, q, r, s and prove that:

(ii) $\frac{1}{\beta\gamma\delta} + \frac{1}{\alpha\gamma\delta} + \frac{1}{\alpha\beta\delta} + \frac{1}{\alpha\beta\gamma} = \frac{1}{qrs} + \frac{1}{prs} + \frac{1}{pqs} + \frac{1}{pqr} - \frac{3}{pqrs}$;

(iii) $\frac{p^2}{(p-\alpha)(p-\beta)(p-\gamma)(p-\delta)} + \frac{q^2}{(q-\alpha)(q-\beta)(q-\gamma)(q-\delta)} + \frac{r^2}{(r-\alpha)(r-\beta)(r-\gamma)(r-\delta)} = 0.$