NSW HSC Mathematics Level 1 (4 Unit) Examination 1969

1. (i) Use Euclid's algorithm to show that 299 and 323 are relatively prime (i.e. their greatest common divisor is 1).

(ii) If a, b, k are integers such that a, k are relatively prime and also b, k are relatively prime prove that ab, k are relatively prime.

2. (i) If m, n are non-negative integers prove that

$$\int_0^{\pi} \cos mx \cos nx \; dx = 0 \text{ if } m \neq n$$

and calculate the value of this integral when $m = n$.

(ii) By applying Euclid's algorithm to $(x^2 + 4)$ and $(x + 1)$ obtain a decomposition of

$$\frac{40}{(x+1)(x^2+4)}$$

into partial fractions and hence show that

$$\int_0^2 \frac{40 \; dx}{(x+1)(x^2+1)} = \pi - 4\log 2 + 8\log 3.$$

3. (i) Find the length of the curve

$$2y = e^x + e^{-x}$$

between $x = 0$ and $x = \log 2$.

(ii) Find the volume of the solid of revolution formed when the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is rotated about the x-axis.

4. By suitably grouping terms in the series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

show that the series is divergent.

By comparison with this series, or otherwise, show that:

(i) $\sum \frac{n}{1+n^2}$ (ii) $\sum \frac{1.3.5.....(2n-1)}{2.4.6.....2n}$
are also divergent.

5. The plane $2x + 3y + 6z = 7$ cuts the square

$$x^2 + y^2 + z^2 = 4$$

in the circle c. Find:

(i) the radius of c;

(ii) the co-ordinates of the centre of c;

(iii) the equation of the cylinder through c with generators parallel to the z-axis.

6. (i) If $a > 0$ and $ab > h^2$ show that the eigenvalues of the matrix

$$\begin{bmatrix} a & h \\ h & b \end{bmatrix} \quad (a, b, h \text{ being real})$$

are real and positive. Hence, assuming the relevant theorem on the reduction of a quadratic form to standard form, deduce that the equation (in plane cartesian co-ordinates)

$$ax^2 + 2hxy + by^2 = 1$$

represents an ellipse of area $\pi/(ab - h^2)^{\frac{1}{2}}$.

[Assume that the area of the ellipse $(x^2/\alpha^2) + (y^2/\beta^2) = 1$ is $\pi\alpha\beta$.]

(ii) A particle P moves in such a way that its cartesian co-ordinates (x, y) at time t are given by

$$x = p\sin(\omega t), \qquad y = q\sin\left(\omega t + \tfrac{\pi}{6}\right)$$

where p, q, ω are positive constants. Show that the path of P is an ellipse of area $\frac{1}{2}\pi pq$.

7. (i) For plane transformations show that the product of two reflections in parallel lines is a displacement and that the product of two reflections in intersecting lines is a rotation.

(ii) A plane p rests upon three fixed points which form a triangle ABC (described anticlockwise). $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ denote anticlockwise rotations of p (in its own plane about A, B, C respectively through angles $2\alpha, 2\beta, 2\gamma$ where α, β, γ denote the angles of the triangle at A, B, C respectively. If

$$\mathcal{U} = \mathcal{T}_1\mathcal{T}_2 \text{ and } \mathcal{V} = \mathcal{T}_1\mathcal{T}_2\mathcal{T}_3$$

describe the geometric transformations represented by $\mathcal{U}$ and $\mathcal{V}$. Also find all the points of p which remain invariant under the transformation $\mathcal{V}$.

8. Explain how the definite integral $\int_a^b f(x)\ dx$, for an increasing function f, is defined by dividing the interval $[a, b]$ into sub-intervals and considering areas of "inner" and "outer" rectangles with bases on these sub-intervals.

Illustrate the above when $f(x) = \log x$ with a sub-division of $[1, 2]$ at the points

$$1, r, r^2, r^3, r^4, \ldots, r^n \quad (\text{where } r^n = 2)$$

showing that the sum of the areas of the outer rectangles is

$$\log r \left(2n - \frac{1}{r-1} \right).$$

Deduce that $\int_1^2 \log x \; dx = 2 \log 2 - 1$.

9. Explain how complex numbers are represented on the Argand diagram.

The three roots of the equation

$$x^3 + ax^2 + bx + c = 0$$

(where a, b, c are given complex numbers) are represented on the Argand diagram by the points A, B, C. Prove that ABC is an equilateral triangle if and only if $a^2 = 3b$.

10. A gun fires a shot from O with initial speed V at an angle α with the horizontal. If the acceleration due to gravity is constant ($= g$) prove that the shot describes a parabola of focal length $V^2 \cos^2 \alpha/(2g)$.

If the initial speed V is fixed but the direction of firing can be varied prove that the region of vulnerability (i.e. the set of points that can be hit) consists of points within and on the paraboloid whose equation (referred to a cartesian x, y, z-frame with origin at O and z-axis vertically upwards) is

$$x^2 + y^2 + (2V^2/g)z = V^4/g^2.$$