NSW HSC Mathematics Level 1 (4 Unit) Examination 1968

1. Using Euclid's algorithm, or otherwise, find the highest common factor, $h(x)$ of $x^7 + 1$ and $x^5 + 1$.

By reversing the steps in the algorithm, or otherwise, find the polynomials, $f(x)$ and $g(x)$, of the lowest degree, such that

$$(x^7 + 1)f(x) + (x^5 + 1)g(x) \equiv h(x)$$

2. (i) The complex number w is such that w =2, $\arg w = \frac{1}{6}\pi$. Mark approximately on an Argand diagram (not on graph paper) the following points:

A corresponding to w, B to w^2, C to $w^2 - w$, D to $\overline{w}$, E to $w\overline{w}$.

(ii) Sketch on an Argand diagram the locus of the point P which satisfies the equation

$$|z - 1| = 1.$$

(iii) From the diagram and using Euclidean properties of the locus, or otherwise, show that the point P of part (ii) also satisfies the condition

$$\arg(z - 1) = \arg z^2.$$

What is the complete locus of a point satisfying this condition?

3. (i) By suitably grouping the terms in the series

$$1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \cdots + \frac{1}{n^\alpha} + \cdots,$$

or otherwise, prove that when $\alpha > 1$ the series converges, and when $\alpha \leqslant 1$ the series diverges.
(You may assume that $1 + g + g^2 + \cdots + g^n + \cdots$ converges when $-1 < g < 1$.)

(ii) Sequences u_n, v_n are defined by

$$u_n = \frac{2n}{2n + 100}, v_n = \frac{2n - 100}{2n}$$

Discuss the convergence of the two series:

(a) $\sum u_n$; and

(b) $\sum(u_n - v_n)$.

4. Prove that a necessary and sufficient condition that the three planes

$$\begin{aligned} lx + my + nz &= 0 \\ l'x + m'y + n'z &= 0 \\ l''x + m''y + n''z &= 0 \end{aligned}$$

should have a common line is

$$l''(mm' - m'n) + m''(nl' - n'l) + n''(lm' - l'm) = 0.$$

Show that for three and only three values of a the three planes

$$\begin{aligned} ax - y + z &= 0 \\ x - (a + \frac{1}{2})y &= 0 \\ x + (a - \frac{1}{2})z &= 0 \end{aligned}$$

have a common line.

Find the direction cosines of each of the three common lines and prove that these lines are mutually perpendicular.

5. (i) By transferring the origin to a suitable point (p, q) the equation

$$7x^2 + 12xy - 2y^2 - 2x - 16y - 12 = 0$$

is reduced to the form

$$ax^2 + 2hxy + by^2 = 1.$$

Find p, q, a, h and b.

(ii) $\mathcal{S}_a, \mathcal{S}_b, \mathcal{S}_c$ are the operations of reflection in the lines a, b, c, respectively, where a, b, c contain the sides of an equilateral triangle. Draw a diagram (not on graph paper) which shows the lines e and f, and a segment determining the displacement $\mathcal{D}$, which are such that:

(a) $\mathcal{S}_a\mathcal{S}_b\mathcal{S}_a = \mathcal{S}_e$.

(b) $\mathcal{S}_b\mathcal{S}_a\mathcal{S}_b = \mathcal{S}_f$.

(c) $\mathcal{S}_c\mathcal{S}_b\mathcal{S}_a = \mathcal{D}\mathcal{S}_b$.

6. S is a set of elements $\{A, B, \dots\}$ and $A \times B$ is an object defined from A and B by the operation $\times$.
State sufficient conditions for the set S under the operation $\times$ to form a group.
The matrices $\mathbf{J}, \mathbf{K}$ are

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \mathbf{K} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \text{ matrix } i^2 = -1.$$

Prove that the set of eight matrices $\{\pm\mathbf{1}, \pm\mathbf{J}, \pm\mathbf{K}, \pm\mathbf{JK}\}$ forms a group under the operation of matrix multiplication.

7. Prove that the curve

$$\left(\frac{x}{u}\right)^{1/2} + \left(\frac{y}{v}\right)^{1/2} = 1, u > 0, v > 0,$$

touches the axes Ox and Oy, say at U and V respectively.

Find the area bounded by the axes and the arc of the curve between U and V.

Prove that the curve represented by the rationalized form of the equation above is a parabola. What part of th curve is represented by the printed (irrational) form of the equation?

8. A function $K(x)$ is defined, over the domain $x \geqslant 0$, by the relations

$$\frac{dK(x)}{dx} = -\frac{1}{x} \text{ and } K(1) = 0.$$

(i) Prove that for any positive number a and any number c, and for any positive value of x:

(a) $\frac{dK(ax)}{dx} = -\frac{1}{x}$;

(b) $K(a) + K(x) = K(ax)$;

(c) $K(x^c) = cK(x)$.

(ii) Prove that there is one and only one value h such that $K(h) = 1$, and that h has the property $h^{K(x)} = x$.

9. A, B are the two points where a line in $z = 0$ parallel to Oy meets the parabolic cylinder $y^2 = x$. A, B, C, D are the vertices, in order, of a square in a plane perpendicular to Ox, and the z-coordinates of C and D are positive. A solid block has as its base the region in $z = 0$ bounded by the parabola $y = x^2$ and the line $x = 1$; its sections by planes perpendicular to Ox are squares such as $ABCD$.

(i) Find the equation of the section of this block by the plane $y = 0$.

(ii) The block has two plane faces, one in $z = 0$ and one in $x = 1$. Prove that the remaining part of the surface of the block consists of regions on two papabolic cylinders.

(iii) Find the volume of the block.

10. A particle of unit mass is projected vertically upwards against a constant gravitational force g and a resistance v/c, where v is the velocity of the particle and c is

a constant. s is the distance travelled in time t; at $t = 0, s = 0$, and $v = c(h - g)$ where h is a constant. Write down the equation of motion of the particle.

Find the time taken by the particle to reach its highest point, and find the height of that point.

The particle falls to its original position under gravity and under the same law of resistance. Will the time of descent be greater or less than the time of ascent? Give reasons for your answer.