NSW HSC Mathematics Level 1 (4 Unit) Examination 1967

1. Given that $a > 1$, prove the theorems in the following sequence:

(i) If m is a positive integer, then

$$\frac{a^m + a^{m-1} + \cdots + 1}{m+1} > \frac{a^{m-1} + a^{m-2} + \cdots + 1}{m}.$$

(ii) If m and n are positive integers and $n > m$, then

$$\frac{1}{n}(a^n - 1) > \frac{1}{m}(a^m - 1).$$

(iii) If r and s are positive rationals and $r > s$, then

$$\frac{1}{r}(a^r - 1) > \frac{1}{s}(a^s - 1).$$

2. Express

$$\frac{1 - abx^2}{(1-ax)(1-bx)}$$

in the form $l + \frac{m}{1-ax} + \frac{n}{1-bx}$, where l, m, n are constants.
Given that $R_n(x)$ is a polynomial and that

$$1 - abx^2 \equiv (1-ax)(1-bx)(1 + u_1x + u_2x^2 + \cdots + u_rx^r + \cdots + u_nx^n) + x^{n+1}R_n(x)$$

find u_r and $R_n(x)$.

3. (i) A variable point P on the Argand diagram represents the complex number z. α and ρ are fixed complex numbers. Describe and illustrate with rough sketches the geomtrical relations between the point:

(a) P_1 representing $z + \alpha$, and the point P;

(b) P_2 representing ρz, and the point P;

(c) P_3 representing ρz, and the point P_2.

(ii) A sequence of points, $z_0, z_1, z_2, \ldots$ is defined from an arbitrary point z_0 by the transformations

$$z_n = c\bar{z}_{n-1} + c - 1,$$

where $c = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$.

Determine whether the sequence consists of repetitions of a finite set of points, and if so how many points there are in the set.

4. In a plane, in relation to a given co-ordinate system with the origin O, P_i is the point represented by the column-vector $\mathbf{r}_i$. Let $\mathbf{r}' = \mathbf{Mr}$ be the equation of an affine transformation of the plane onto itself in which P_i is transformed into P_i'.

Prove that:

(i) if P_1, P_2, P_3 are collilear, then

(a) P_1', P_2', P_3' are collinear, and

(b) $*P_1'P_2'/*P_1'P_3' = *P_1P_2/*P_1P_3$;

(ii) area $\Delta OP_1'P_2' = (\text{area } \Delta OP_1P_2).\det \mathbf{M}$.

5. (i) Prove that if, for two 2×2 matrices, $\mathbf{M}, \mathbf{N}$, a matrix $\mathbf{C}$ exists such that

$$\mathbf{N} = \mathbf{CMC}^{-1},$$

then the two quadratic polynomials $\det(x\mathbf{1} - \mathbf{M})$ and $\det(x\mathbf{1} - \mathbf{N})$ are identical.

(ii) $\mathcal{D}$ is the operation of displacement through one unit parallel to the x-axis in a rectangular co-ordinate system, origin O; $\mathcal{S}$ is the operation of reflexion in the line $y = x$. Draw a sketch to show the points $\mathcal{D}O, \mathcal{D}^2O, \mathcal{S}\mathcal{D}O, \mathcal{D}\mathcal{S}\mathcal{D}O, \mathcal{D}\mathcal{S}\mathcal{D}\mathcal{S}\mathcal{D}O$.
Prove that, for any sequence of non-negative integers $r, s, t, \ldots,$

$$\ldots \mathcal{D}^t\mathcal{S}\mathcal{D}^s\mathcal{S}\mathcal{D}^rO = \mathcal{D}^m\mathcal{S}\mathcal{D}^nO,$$

where m, n are non-negative integers. Describe the set of points which corresponds to he set of ordered pairs (m, n).

6. Referred to a rectangular co-ordinate system in space, A is $(3, 11, -4)$, B is $(5, -3, 0)$, C is $(0, 2, -5)$.

Find the equations of

(i) the plane through the line OA and perpendicular to the plane OBC;

(ii) the plane through the line OA and parallel to the line BC;

(iii) the circular cylinder, with generators parallel to the y-axis, which passes through A, B, C.

7. (i) Prove that

$$\frac{1}{2}\Big(\frac{a}{x} + x\Big) - \sqrt{a} = \frac{(\sqrt{a} - x)^2}{2x}.$$

(ii) In a certain set of tables, in which the value of $\sqrt{a}$ is given to 4 significant figures, the value of $\sqrt{2}$ is given as $1 \cdot 414$. If $c \leqslant 9$ what is the largest integer n for which

$$\Big|\sqrt{2} - \frac{1}{2}\Big(\frac{2}{1 \cdot 414} + 1 \cdot 414\Big)\Big| < c10^{-n}.$$

Interpret this as a statement of the greatest number of significant figures in the value of $\sqrt{2}$ that can be computed in one application of formula (i) to the given approximation $1 \cdot 414$.

8. c is that part of the curve represented by the equation

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

which lies in the first quadrant ($x \geqslant 0, y \geqslant 0$).

(i) Find the equation of the tangent to c at (x_0, y_0).

(ii) Show that the segment intercepted on the tangent by the co-ordinate axes has constant length.

(iii) If s is the distance along c from some fixed point on c, prove that $\frac{ds}{dx} = \left(\frac{a}{x}\right)^{\frac{1}{3}}$.

(iv) Find the length of c.

9. (i) If $u_n = \int \frac{dx}{(x^2+1)^n}$
show that $u_n - u_{n-1}$ can be expressed in the form

$$\frac{Ax}{(x^2+1)^{n-1}} + Bu_{n-1}$$

where A and B are constants.

(ii) Find $\lim_{k\to\infty} \int_0^k \frac{dx}{(x^2+1)^2}$.

10. A train of mass M, pulled by a locomotive which exerts a tractive force $P(v)$, is moving at speed v along a level track against a resistive force $R(v)$. Both $P(v)$ and $R(v)$ are functions of v.

(i) If $R(v) = a + bv^2$, and $P(v) = 2a$, where a, b are positive constants, prove that there is an upper bound to the speed that the train can attain, and find the value of this upper bound.

(ii) Show that, in accelerating from speed v_0 to speed v_1, the distance travelled by the train is $kM \int_{v_0}^{v_1} \frac{vdv}{P(v)-R(v)}$ where k is a constant.

What is the value of k?