2017 HSC Examination

Mathematics Extension 1

SOLUTIONS

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Multiple Choice

1. A
2. B
3. B
4. C
5. D
6. D
7. A
8. C
9. C
10. B

Brief Explanations

Question 1 Use the factor theorem. We find that $P(2)=0$, so $(x-2)$ is a factor.

Question 2 We have $3\log_a(2)=1.893$, so calculate $\log_a(4)=2\log_a(2)$.

Question 3 $\angle BAD=180^\circ-110^\circ=70^\circ$ (opposite angles in a cyclic quadrilateral) and $\angle TAB=65^\circ$ (alternate segment theorem).

Question 4 Standard auxiliary angle formula problem.

Question 5 As $x\to\infty,\ y\to-2$. The only graph with horizontal asymptote $y=-2$ is (D).

Question 6 The normal is usually $x+py=p^3+2p$ for a point with coordinates $(2p,p^2)$ (set $a=1$ in the formula in the reference sheet). Now replace *p* with $\frac{1}{p}$ and re-arrange.

Question 7 The domain is found by solving $-1\le\frac{3}{x}\le1$. The direction of the arrows can be found quickly by drawing $y=\frac{3}{x}$ and the lines $y=\pm1$.

Question 8 Use the chain rule $\frac{dA}{dt}=\frac{dA}{dr}\times\frac{dr}{dt}=2\pi r\times5$, then substitute in $r=15$.

Question 9 Find the power of *x* of each general term. Option (C) yields $7r-28$, and is actually the only one that yields an integer value of *r*.

Question 10 The total number ways to cross 3 squares out of 9 is $\binom{9}{3}$. Since the grid is small, we can manually count the number of possible ways to get three in a row, which will be 8. Hence, the probability is $\frac{8}{\binom{9}{3}}$.

Written Response

Question 11 (a)

$$x_P = \frac{3(-4)+2(1)}{2+3} = -2$$

Question 11 (b)

$$\begin{aligned}\frac{d}{dx}\left(\tan^{-1}\left(x^3\right)\right) &= \frac{3x^2}{1+\left(x^3\right)^2} \\ &= \frac{3x^2}{1+x^6}\end{aligned}$$

Question 11 (c)

Multiply both sides by $(x+1)^2$.

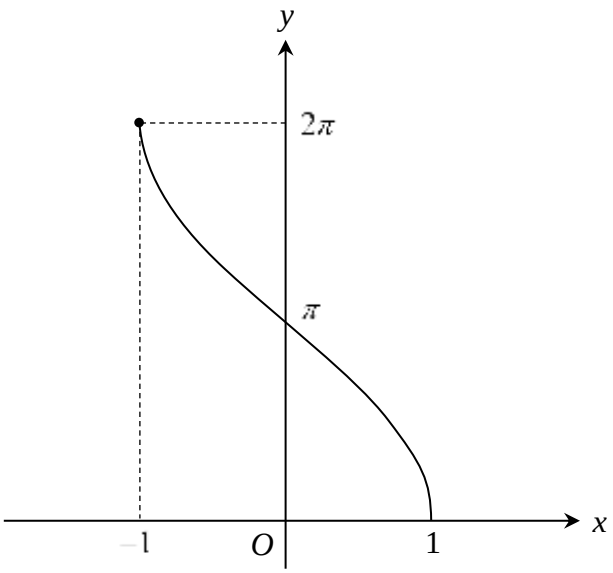
$$\begin{aligned}\frac{2x}{x+1} &> 1 \\ 2x(x+1) &> (x+1)^2 \\ (x+1)^2 - 2x(x+1) &< 0 \\ (x+1)[x+1-2x] &< 0 \\ (x+1)(1-x) &< 0 \\ (x+1)(x-1) &> 0\end{aligned}$$

Hence, $x > 1$ or $x < -1$.

Question 11 (d)

Domain: $-1 \le x \le 1$

Range: $0 \le y \le 2\pi$



Question 11 (e)

Compute the differential term first.

$$x = u^2 - 1$$
$$dx = 2u\,du$$

Compute the new limits.

$$x = 3 \quad \rightarrow \quad u = 2$$
$$x = 0 \quad \rightarrow \quad u = 1$$

Substitute.

$$\begin{aligned}\int_0^3 \frac{x}{\sqrt{x+1}}\,dx &= \int_1^2 \frac{u^2-1}{u} \times 2u\ du \\ &= 2\int_1^2 u^2 - 1\,du \\ &= 2\left[\frac{u^3}{3} - u\right]_1^2 \\ &= \frac{8}{3}\end{aligned}$$

Question 11 (f)

Use the fact that $\int f^n(x)\, f'(x)\, dx = \dfrac{1}{n+1} f^{n+1}(x) + C$.

$$\int \sin^2 x \cos x \, dx = \frac{1}{3}\sin^3 x + C$$

Question 11 (g) (i)

$$P(3 \text{ red}) = \binom{8}{3}\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^5$$

Question 11 (g) (ii)

$$P(\text{no red flowers}) = \left(\frac{4}{5}\right)^8$$

Question 11 (g) (iii)

$$\begin{aligned} P(\text{at least one red}) &= 1 - P(\text{no red}) \\ &= 1 - \left(\frac{4}{5}\right)^8 \end{aligned}$$

Question 12 (a)

Let D be any point on the major arc AC.

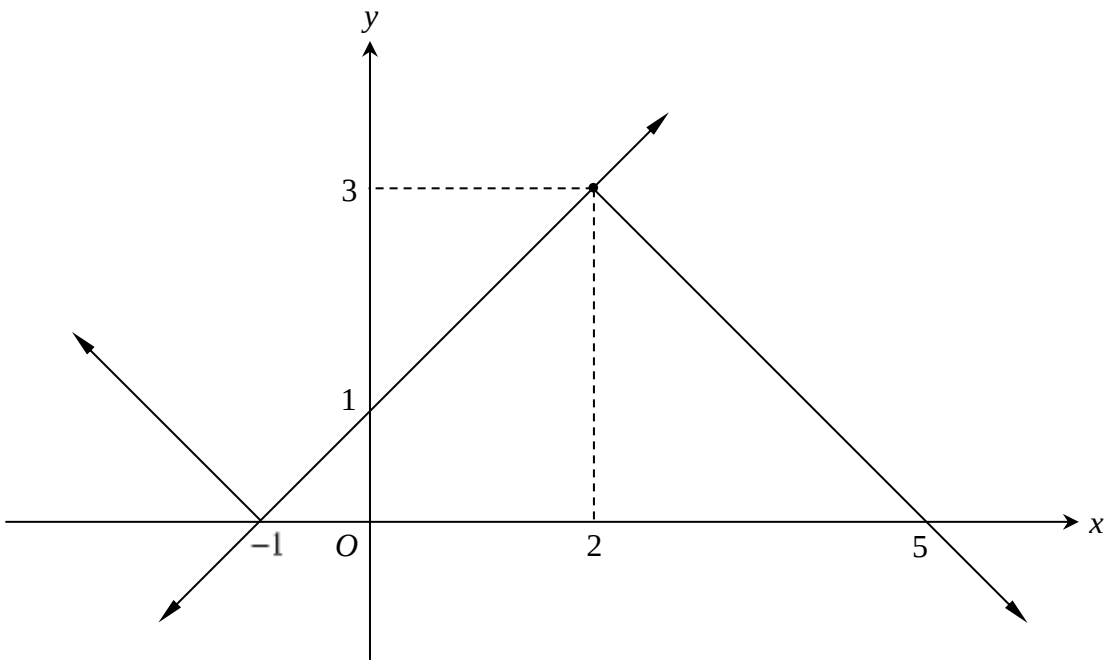
$$\angle ADC = 50^\circ \quad ...(\angle \text{ at centre} = 2 \times \angle \text{at circumference})$$

$$\begin{aligned}\angle ABC &= 180^\circ - 50^\circ \quad ...(\text{opposite angles of a cyclic quad. are supplementary}) \\ &= 130^\circ\end{aligned}$$

Question 12 (b) (i)

The graph of $y = |x+1|$ is the shape of $y = |x|$ but shifted to the left by 1 unit.

The graph of $y = 3 - |x-2|$ is the shape of $y = |x|$ but upside-down, shifted to the right by 1 unit and shifted up by 3 units, so the vertex is at $(2,3)$. But observe that $(2,3)$ lies on the right branch of $y = |x+1|$.



Question 12 (b) (ii)

We are asked to solve $|x+1|+|x-2|=3$, but this is equivalent to $|x+1|=3-|x-2|$, which in turn is equivalent to solving $y=|x+1|$ and $y=3-|x-2|$ simultaneously. Since we have the graphs of the two equations above, we can find the solutions geometrically.

From the diagram above, the two equations are equal over the domain $-1\le x\le 2$, so therefore that must be the solution set to the original equation $|x+1|+|x-2|=3$.

Question 12 (c) (i)

First, find the ratio of the left part to the right part.

$$\begin{aligned}\frac{V_{\text{left}}}{V_{\text{right}}} &= \frac{\pi\displaystyle\int_{-1}^{h} 1-x^2\,dx}{\pi\displaystyle\int_{h}^{1} 1-x^2\,dx}\\ &= \frac{\left[x-\dfrac{x^3}{3}\right]_{-1}^{h}}{\left[x-\dfrac{x^3}{3}\right]_{h}^{1}}\\ &= \frac{h-\dfrac{h^3}{3}+\dfrac{2}{3}}{\dfrac{2}{3}-h+\dfrac{h^3}{3}}\\ &= \frac{3h-h^3+2}{2-3h+h^3}\end{aligned}$$

But the ratio is given to be $2:1$, so this fraction must be equal to 2.

$$\begin{aligned}\frac{3h-h^3+2}{2-3h+h^3} &= 2\\ 3h-h^3+2 &= 2\left(2-3h+h^3\right)\\ &= 4-6h+2h^3\end{aligned}$$

And hence the required result $3h^3-9h+2=0$.

Question 12 (c) (ii)

Define $f(h)=3h^3-9h+2$, so $f'(h)=9h^2-9$.

$$\begin{aligned} h_2 &= h_1 - \frac{f(h_1)}{f'(h_1)} \\ &= h_1 - \frac{3h_1^{\,3}-9h_1+2}{9h_1^{\,2}-9} \\ &= \frac{2}{9} \qquad \ldots(\text{since } h_1 = 0) \end{aligned}$$

Question 12 (d)

Differentiate *t* with respect to *x* and notice that $\frac{dt}{dx}=\frac{1}{dx/dt}=\frac{1}{v}$.

$$\begin{aligned} \frac{dt}{dx} &= 2e^{-2x} \\ \frac{1}{v} &= 2e^{-2x} \\ v &= \frac{1}{2}e^{2x} \\ v^2 &= \frac{1}{4}e^{4x} \end{aligned}$$

But $a=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

$$\begin{aligned} a &= \frac{d}{dx}\left(\frac{1}{8}e^{4x}\right) \\ &= \frac{1}{2}e^{4x} \end{aligned}$$

Question 12 (e)

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1-\cos 2x}{x^2}\right) &= \lim_{x\to 0}\left(\frac{2\sin^2 x}{x^2}\right)\\ &= 2\times\lim_{x\to 0}\left(\frac{\sin x}{x}\times\frac{\sin x}{x}\right)\\ &= 2\times\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\times\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\\ &= 2\times 1\times 1\\ &= 2\end{aligned}$$

Question 13 (a)

The particle satisfies the equation for simple harmonic motion $v^2 = n^2\left(a^2 - x^2\right)$.

Substitute in the given data and solve simultaneously.

$$\begin{aligned} x = 2,\ v = 4 \quad &\rightarrow \quad 16 = n^2\left(a^2 - 4\right)\\ x = 5,\ v = 3 \quad &\rightarrow \quad 9 = n^2\left(a^2 - 25\right)\end{aligned}$$

Divide these equations.

$$\begin{aligned}\frac{16}{9} &= \frac{a^2-4}{a^2-25}\\ 16a^2 - 400 &= 9a^2 - 36\\ 7a^2 &= 364\\ a^2 &= 52\end{aligned}$$

Substitute back in to find *n*.

$$\begin{aligned}9 &= n^2\left(52-25\right)\\ n^2 &= \frac{9}{27}\\ &= \frac{1}{3}\\ n &= \frac{1}{\sqrt{3}}\end{aligned}$$

Hence, the period is $T = \dfrac{2\pi}{n} = 2\pi\sqrt{3}$.

Question 13 (b) (i)

$$(1+x)^n+(1-x)^n=\left[\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\ldots+\binom{n}{n}x^n\right]+\left[\binom{n}{0}-\binom{n}{1}x+\binom{n}{2}x^2-\ldots+(-1)^n\binom{n}{n}x^n\right]$$

Notice that all the odd powered terms cancel and that we have two of each even powered term. Furthermore, the last term does not cancel because n is even.

Hence, we have the expansion

$$\begin{aligned}(1+x)^n+(1-x)^n&=2\binom{n}{0}+2\binom{n}{2}x^2+2\binom{n}{4}x^4+\ldots+2\binom{n}{n}x^n\\&=2\left[\binom{n}{0}+\binom{n}{2}x^2+\binom{n}{4}x^4+\ldots+\binom{n}{n}x^n\right]\end{aligned}$$

Question 13 (b) (ii)

Differentiate both sides of (i) with respect to x.

$$n(1+x)^{n-1}-n(1-x)^{n-1}=2\left[2\binom{n}{2}x+4\binom{n}{4}x^3\ldots+n\binom{n}{n}x^{n-1}\right]$$

Question 13 (b) (iii)

Let $x=1$ and then re-arrange.

$$\begin{aligned}2\left[2\binom{n}{2}+4\binom{n}{4}\ldots+n\binom{n}{n}\right]&=n2^{n-1}\\2\binom{n}{2}+4\binom{n}{4}\ldots+n\binom{n}{n}&=n2^{n-2}\\\binom{n}{2}+2\binom{n}{4}\ldots+\frac{n}{2}\binom{n}{n}&=n2^{n-3}\qquad\ldots(\text{divide both sides by }2)\end{aligned}$$

Question 13 (c) (i)

Let $y = 0$ to find the time of flight.

$$Vt\sin\theta - \frac{1}{2}gt^2 = 0$$
$$V\sin\theta - \frac{1}{2}gt = 0$$
$$t = \frac{2V\sin\theta}{g}$$

Substitute back into x to find horizontal range.

$$x = V\left(\frac{2V\sin\theta}{g}\right)\cos\theta$$
$$= \frac{2V^2\sin\theta\cos\theta}{g}$$
$$= \frac{V^2\sin 2\theta}{g}$$

Question 13 (c) (ii)

Substitute the given inequality for V^2 into (i).

$$x = \frac{V^2\sin 2\theta}{g}$$
$$< \frac{100g\sin 2\theta}{g}$$
$$= 100\sin 2\theta$$
$$\leq 100 \quad ...(\text{since } \sin 2\theta \leq 1)$$

Hence, we have the bound $x < 100$.

Question 13 (c) (iii)

We are given that

$$\frac{V^2 \sin 2\theta}{g} \geq 100$$

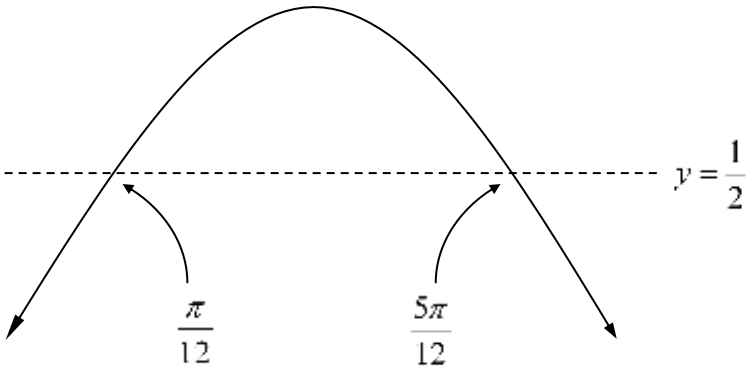
Let $V^2 = 200g$.

$$\frac{200g \sin 2\theta}{g} \geq 100$$
$$200 \sin 2\theta \geq 100$$
$$\sin 2\theta \geq \frac{1}{2}$$

The boundary of the inequality is obtained by solving $\sin 2\theta = \frac{1}{2}$.

$$\sin 2\theta = \frac{1}{2}$$
$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$
$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

A rough sketch shows that the values of θ must lie *between* these two boundaries, since we want the sine curve to be *above* the line $y = \frac{1}{2}$. Hence, we have the result $\frac{\pi}{12} \leq \theta \leq \frac{5\pi}{12}$



Question 13 (c) (iv)

The greatest height is achieved when $\dot{y} = 0$.

$$V \sin\theta - gt = 0$$
$$t = \frac{V \sin\theta}{g}$$

For the actual maximum height, substitute this into y.

$$\begin{aligned} y_{\max} &= Vt\sin\theta - \frac{1}{2}gt^2 \\ &= V\left(\frac{V\sin\theta}{g}\right)\sin\theta - \frac{1}{2}g\left(\frac{V\sin\theta}{g}\right)^2 \\ &= \frac{V^2\sin^2\theta}{g} - \frac{V^2\sin^2\theta}{2g} \\ &= \frac{V^2\sin^2\theta}{2g} \\ &= \frac{200g\sin^2\theta}{2g} \qquad \ldots\left(\text{since } V^2 = 200g\right) \\ &= 100\sin^2\theta \end{aligned}$$

But note that $\frac{\pi}{12} \le \theta \le \frac{5\pi}{12}$, so the maximum height is attained when $\theta = \frac{5\pi}{12}$ (the curve is still increasing at this stage and has not yet dipped down).

Hence, the maximum height is

$$\begin{aligned} y_{\max} &= 100\sin^2\left(\frac{5\pi}{12}\right) \\ &= 50 - 50\cos\left(\frac{5\pi}{6}\right) \\ &= 50 + 25\sqrt{3} \end{aligned}$$

NOTE: Given that this part was only worth 2 marks, it most likely wasn't necessary to go as far as finding a nice exact form for $y_{\max}$. Full marks most likely awarded for finding that $y_{\max} = 100\sin^2\left(\frac{5\pi}{12}\right)$.

Question 14 (a)

Base Case: $n=1$

$$8^3+6^1=518=7\times 74$$

Hence, the result is true for $n=1$.

Inductive Hypothesis: $n=k$

$$8^{2k+1}+6^{2k-1}=7M \qquad ...(\text{where } M \text{ is an integer})$$

Inductive Step: $n=k \Rightarrow n=k+1$

$$\begin{aligned}
8^{2(k+1)+1}+6^{2(k+1)-1} &= 8^{2k+3}+6^{2k+1} \\
&= 8^2\times 8^{2k+1}+6^2\times 6^{2k-1} \\
&= 8^2\times\left(7M-6^{2k-1}\right)+6^2\times 6^{2k-1} \\
&= 64\times 7M-64\left(6^{2k-1}\right)+36\left(6^{2k-1}\right) \\
&= 64\times 7M-28\left(6^{2k-1}\right) \\
&= 7\underbrace{\left[64M-4\left(6^{2k-1}\right)\right]}_{\text{integer}}
\end{aligned}$$

And hence, the result is true by the principle of mathematical induction.

Question 14 (b) (i)

The equation of the tangent from *P* is $y=px-p^2$ (reference sheet), noting that $a=1$. Solve simultaneously with the bottom parabola $x^2=-4ay$.

$$\begin{aligned}
x^2 &= -4a\left(px-p^2\right) \\
&= -4apx+4ap^2 \\
x^2+4apx-4ap^2 &= 0
\end{aligned}$$

Question 14 (b) (ii)

Note that the roots of the equation in (i) are the x coordinates of where the tangent intersects Q and R.

$$\begin{aligned} x_M &= \frac{x_R + x_Q}{2} \\ &= \frac{\text{Sum of roots of the quadratic in (i)}}{2} \\ &= \frac{-4ap}{2} \\ &= -2ap \end{aligned}$$

Substitute back into the linear function to find the y coordinate.

$$\begin{aligned} y_M &= p(-2ap) - p^2 \\ &= -2ap^2 - p^2 \\ &= -p^2(2a+1) \end{aligned}$$

Hence, M has coordinates $M\left(-2ap, -p^2(2a+1)\right)$.

Question 14 (b) (iii)

If M lies on the parabola $x^2 = -4y$, then it must satisfy that equation. So we substitute the coordinates of M into the equation, then solve for a.

$$\begin{aligned} (-2ap)^2 &= -4\left(-p^2(2a+1)\right) \\ 4a^2p^2 &= 4p^2(2a+1) \\ a^2 &= 2a+1 \\ a^2 - 2a - 1 &= 0 \\ a &= 1 \pm \sqrt{2} \end{aligned}$$

Since $a > 0$, the value of a is $a = 1 + \sqrt{2}$.

Question 14 (c) (i)

$$\begin{aligned}\frac{d}{dt}\left(F(t)e^{0.4t}\right) &= F'(t)e^{0.4t} + 0.4e^{0.4t}F(t) \\ &= \left(50e^{-0.5t} - 0.4F(t)\right)e^{0.4t} + 0.4e^{0.4t}F(t) \\ &= 50e^{-0.1t} - 0.4e^{0.4t}F(t) + 0.4e^{0.4t}F(t) \\ &= 50e^{-0.1t}\end{aligned}$$

Question 14 (c) (ii)

Integrate the result from (i) with respect to *t*.

$$F(t)e^{0.4t} = -500e^{-0.1t} + C$$

Substitute in the fact that $F(0) = 0$ (given).

$$\begin{aligned}0 \times e^{0} &= -500e^{0} + C \\ C &= 500\end{aligned}$$

Hence

$$\begin{aligned}F(t)e^{0.4t} &= -500e^{-0.1t} + 500 \\ F(t) &= -500e^{-0.5t} + 500e^{-0.4t} \\ &= 500\left(e^{-0.4t} - e^{-0.5t}\right)\end{aligned}$$

Question 14 (c) (iii)

$$\begin{aligned}F'(t) &= 500\left(-0.4e^{-0.4t} + 0.5e^{-0.5t}\right) \\ &= -200e^{-0.4t} + 250e^{-0.5t} \\ &= 0 \qquad ...(\text{for max/min value of } F)\end{aligned}$$

Solve for *t*.

$$\begin{aligned}200e^{-0.4t} &= 250e^{-0.5t} \\ 200e^{0.1t} &= 250 \\ e^{0.1t} &= \frac{5}{4} \\ t &= 10\ln\left(\frac{5}{4}\right)\end{aligned}$$