2016 HSC Examination

Mathematics Extension 1

SOLUTIONS

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Multiple Choice

1. D
2. B
3. A
4. C
5. A
6. D
7. C
8. B
9. A
10. A

Brief Explanations

Question 1 Just expand the sigma notation.

Question 2 The remainder is given by substituting in $x=2$ into the cubic.

Question 3 The given expression is the expansion of $\tan(2x-x)=\tan x$.

Question 4 *TDOA* is cyclic because $\angle ODT=\angle OAT=90^\circ$. Hence $\angle DOA_{\text{obtuse}}=140^\circ$. So the reflex angle is $360^\circ-140^\circ=220^\circ$.

Question 5 Use the fact that $\sin^2 2x=\frac{1}{2}(1-\cos 4x)$.

Question 6 Factorise to get $(2\sin x-1)(\sin x-3)=0$. But $\sin x\neq 3$, so $\sin x=\frac{1}{2}$.

Question 7 Differentiate to get $\dot{x}=20\cos(4t)-48\sin(4t)=A\cos(\ldots\text{who cares}\ldots)$, as we would proceed for auxiliary angle formulae. Here, $A=\sqrt{20^2+48^2}=52$, so $\dot{x}_{\max}=52$.

Question 8 We consider the total $\binom{18}{11}$, then remove the teams where nobody is left handed, which there are $\binom{15}{11}$ of. Hence, the answer is $\binom{18}{11}-\binom{15}{11}=30459$.

Question 9 From the diagram, we can clearly see that $f'(1)>1$. It is not truly clear, but we can see that $f''(1)$ is either negative (concave down) or possibly near zero (close to the point of inflexion). In either case, the answer must be (A).

Question 10 Since $a>0$, it is either (A) or (B). Since $a,\ b>0$, the sum of roots must be negative, and so the answer is (A).

Written Response

Question 11 (a)

Swap x and y.

$$\begin{aligned} x &= y^3 - 2 \\ y^3 &= x + 2 \\ y &= \sqrt[3]{x+2} \end{aligned}$$

Question 11 (b)

Let $u = x - 4$, so $du = dx$.

$$\begin{aligned} I &= \int x\sqrt{x-4}\, dx \\ &= \int (u+4)\sqrt{u}\, du \\ &= \int u^{\frac{3}{2}} + 4u^{\frac{1}{2}}\, du \\ &= \frac{2}{5}u^{\frac{5}{2}} + \frac{8}{3}u^{\frac{3}{2}} + C \\ &= \frac{2}{5}(x-4)^{\frac{5}{2}} + \frac{8}{3}(x-4)^{\frac{3}{2}} + C \end{aligned}$$

Question 11 (c)

Use the chain rule.

$$\begin{aligned} \frac{d}{dx}\left(3\tan^{-1}(2x)\right) &= 3 \times \frac{2}{1+(2x)^2} \\ &= \frac{6}{1+4x^2} \end{aligned}$$

Question 11 (d)

Note the double angle formula $2\sin x\cos x = \sin 2x$.

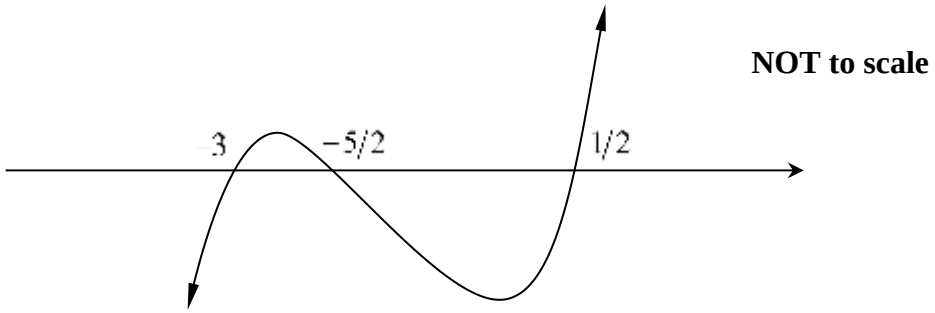
$$\begin{aligned}\lim_{x\to 0}\left(\frac{2\sin x\cos x}{3x}\right) &= \lim_{x\to 0}\left(\frac{\sin 2x}{3x}\right)\\ &= \frac{2}{3}\times\lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)\\ &= \frac{2}{3}\end{aligned}$$

Question 11 (e)

Multiply both sides by $(2x+5)^2$.

$$\begin{aligned}3(2x+5)-x(2x+5)^2 &> 0\\ (2x+5)\left[3-x(2x+5)\right] &> 0\\ (2x+5)\left[-2x^2-5x+3\right] &> 0\\ (2x+5)(2x^2+5x-3) &< 0\\ (2x+5)(2x-1)(x+3) &< 0\end{aligned}$$

This is a positive cubic with roots (in increasing order) $x = -3, -\frac{5}{2}, \frac{1}{2}$.



From the sketch, we see that the cubic has negative *y* values when $x < -3$ or $-\frac{5}{2} < x < \frac{1}{2}$.

Question 11 (f) (i)

Standard binomial probability question.

$$\binom{3}{1}\left(\frac{3}{5}\right)^1\left(\frac{2}{5}\right)^2=\frac{36}{125}$$

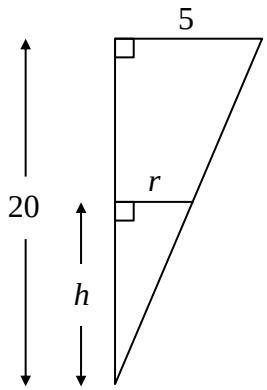
Question 11 (f) (ii)

We consider the complementary event, and take it away from the total.

$$\begin{aligned}P\left(\text{bullseye at least 2 times out of 6}\right)&=1-P\left(0\text{ bullseyes}\right)-P\left(1\text{ bullseyes}\right)\\&=1-\binom{6}{0}\left(\frac{3}{5}\right)^0\left(\frac{2}{5}\right)^6-\binom{6}{1}\left(\frac{3}{5}\right)^1\left(\frac{2}{5}\right)^5\\&=\frac{2997}{3125}\end{aligned}$$

Question 12 (a) (i)

Consider the frontal cross section.



Using similar triangle ratios, we have $\frac{r}{h}=\frac{5}{20}=\frac{1}{4}$.

Question 12 (a) (ii)

First, find the volume in terms of *h*.

$$\begin{aligned} v &= \frac{\pi}{3} r^2 h \\ &= \frac{\pi}{3}\left(\frac{h}{4}\right)^2 h \\ &= \frac{\pi}{48} h^3 \end{aligned}$$

Now we can differentiate with respect to *h*.

$$\frac{dv}{dh} = 3 \times \frac{\pi}{48} h^2 = \frac{\pi}{16} h^2$$

Question 12 (a) (iii)

Use the chain rule.

$$\begin{aligned} \frac{dh}{dt} &= \frac{dh}{dA} \times \frac{dA}{dt} \\ &= \frac{dh}{dA} \times (-0.04) \end{aligned}$$

But $A = \pi r^2 = \frac{\pi}{16} h^2$, since $r = \frac{h}{4}$.

Hence

$$\begin{aligned} \frac{dh}{dt} &= \frac{dh}{dA} \times (-0.04) \\ &= \frac{1}{dA/dh} \times (-0.04) \\ &= \frac{1}{\pi h/8} \times (-0.04) \\ &= \frac{-0.32}{\pi h} \end{aligned}$$

Question 12 (a) (iv)

Again, use the chain rule.

$$\begin{aligned}\frac{dv}{dt} &= \frac{dv}{dh} \times \frac{dh}{dt} \\ &= \underset{\text{part (ii)}}{\frac{\pi}{16}h^2} \times \underset{\text{part (iii)}}{\frac{-0.32}{\pi h}} \\ &= -\frac{h}{50}\end{aligned}$$

So when $h = 10$, $\dfrac{dv}{dt} = -0.2\text{cm}^3/\text{s}$.

Question 12 (b) (i)

Since the rate that X increases is proportional to the mass of Y, we have

$$\begin{aligned}\frac{dx}{dt} &= ky \\ &= k(500 - x) \quad ...(\text{since } x + y = 500)\end{aligned}$$

Initially, $x = 0$ since the reaction has not begun yet, so substituting this in, we find $k = \dfrac{1}{250} = 0.004$, and hence $\dfrac{dx}{dt} = 0.004(500 - x)$.

Question 12 (b) (ii)

Differentiating the given equation $x = 500 - Ae^{-0.004t}$, we have

$$\begin{aligned}\frac{dx}{dt} &= 0.004Ae^{-0.004t} \\ &= 0.004(500 - x)\end{aligned}$$

And so it satisfies the equation in part (i).

We also want $x(0) = 0$, so we must have $A = 500$.

Question 12 (c) (i)

Let m_1 be the gradient of the tangent to the $y = \tan x$ at $x = \alpha$, and let m_2 be similarly defined for $y = \cos x$.

By differentiating the equations and substituting in $x = \alpha$, we have

$$m_1 = \sec^2 \alpha$$
$$m_2 = -\sin \alpha$$

Compute the product.

$$\begin{aligned} m_1 m_2 &= -\sin \alpha \sec^2 \alpha \\ &= -\frac{\tan \alpha}{\cos \alpha} \\ &= -1 \qquad \ldots(\text{since } \tan x = \cos x \text{ at } x = \alpha) \end{aligned}$$

Question 12 (c) (ii)

Let $f(x) = \tan x - \cos x$, which has the solution $x = \alpha$ since $\tan \alpha = \cos \alpha$.

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= 1 - \frac{\tan(1) - \cos(1)}{\sec^2(1) + \sin(1)} \qquad \ldots(\text{calculator in radian mode}) \\ &\approx 0.76 \end{aligned}$$

Question 13 (a) (i)

Since the distance between the high tide and low tide is 8 metres, the amplitude must be 4, and the centre of motion $x = 5$.

We know that a period is 12.5 hours, so $\frac{2\pi}{n} = 12.5 \quad \Rightarrow \quad n = \frac{4\pi}{25}$.

Since we are beginning from high tide (the positive extremity), we use the cosine function.

Hence $x = 5 + 4\cos\left(\frac{4\pi}{25} t\right)$ is an appropriate model for the tides.

Question 13 (a) (ii)

The tide increases at the fastest rate when the water level is at the centre of motion $x=5$. But we in fact want the *second* time this occurs. This is because during the first time this occurs, the water level is *decreasing*.

$$\begin{aligned}
5+4\cos\left(\frac{4\pi}{25}t\right)&=5\\
\cos\left(\frac{4\pi}{25}t\right)&=0\\
\frac{4\pi}{25}t&=\frac{3\pi}{2} \qquad \ldots\left(\text{the first solution would have been } \frac{\pi}{2}\right)\\
t&=\frac{75}{8}\\
&\approx 9 \text{ hours, } 22 \text{ minutes, } 30 \text{ seconds}
\end{aligned}$$

So the earliest time is 9 hours, 22 minutes and 30 seconds from 2am, which is $11:22.30\text{am}$.

Question 13 (b) (i)

The greatest height occurs when $\dot{y}=u\sin\theta-10t=0$, which yields $t=\dfrac{u\sin\theta}{10}$.

Substitute this back into $y=ut\sin\theta-5t^2$.

$$\begin{aligned}
y&=u\left(\frac{u\sin\theta}{10}\right)\sin\theta-5\left(\frac{u\sin\theta}{10}\right)^2\\
&=\frac{u^2\sin^2\theta}{10}-\frac{u^2\sin^2\theta}{20}\\
&=\frac{u^2\sin^2\theta}{20}
\end{aligned}$$

Question 13 (b) (ii)

Let $\theta = 30^\circ$, $u = 30\,\text{ms}^{-1}$ and add 20 metres to our answer to take into account the fact that the ball is initially 20m above the ground.

$$\begin{aligned} h &= 20 + \frac{u^2 \sin^2 \theta}{20} \\ &= 20 + \frac{900\left(1/4\right)}{20} \\ &= \frac{125}{4} \end{aligned}$$

Question 13 (b) (iii)

This is basically another projectile motion, but with initial angle zero, initial velocity $10\,\text{ms}^{-1}$ and initial height $\frac{125}{4}$ metres.

The vertical of motion is $y = \frac{125}{4} - 5t^2$. Let this be zero to find the time of flight.

$$\begin{aligned} \frac{125}{4} - 5t^2 &= 0 \\ t^2 &= \frac{25}{4} \\ t &= \frac{5}{2}\text{seconds} \quad ...\left(\text{taking } t > 0\right) \end{aligned}$$

Question 13 (b) (iii)

The horizontal equation of motion is $x = 10t$, and so the horizontal range after rebounding is simply $x = 10\left(\frac{5}{2}\right) = 25$ metres.

Question 13 (c) (i)

$\angle ACB = \angle ADB = 90^\circ$ (angle in a semi-circle)

$\angle MCE = \angle MDE = 90^\circ$ (supplementary angles in a straight line)

Since $\angle MCE$ and $\angle MDE$ are supplementary and opposite, it follows that $CMDE$ is cyclic.

Question 13 (c) (ii)

Since *CMDE* is cyclic, we have $\angle CME = \angle CDE$ (angles subtended by arc *CE*).

Similarly, $\angle CDE = \angle CBA$ (angles subtended by arc *AC*)

Hence *CMBF* is cyclic since $\angle CMF = \angle CBF$ (converse of the above theorems).

Therefore $\angle MFB = \angle MCB = 90^\circ$ (angles subtended by arc *BM*), and so $MF \perp AB$.

Question 14 (a) (i)

Expand the given product.

$$\begin{aligned}(n+1)\left(4n^2+14n+9\right) &= 4n^3+14n^2+9n+4n^2+14n+9 \\ &= 4n^3+18n^2+23n+9\end{aligned}$$

Question 14 (a) (ii)

Base Case: $n=1$

$\text{LHS} = 1\times 3 = 3 \qquad \text{RHS} = \frac{1}{3}(4+6-1) = 3$

Inductive Hypothesis: $n=k$

$1\times 3+3\times 5+5\times 7+\ldots+(2k-1)(2k+1) = \frac{k}{3}\left(4k^2+6k-1\right)$, where $k \geq 1$

Inductive Step: $n=k \Rightarrow n=k+1$

RTP : $\underbrace{1\times 3+3\times 5+5\times 7+\ldots+(2k-1)(2k+1)}_{n=k}+(2k+1)(2k+3) = \frac{k+1}{3}\left(4k^2+14k+9\right)$

$$\begin{aligned}\text{LHS} &= 1\times 3+3\times 5+5\times 7+\ldots+(2k-1)(2k+1)+(2k+1)(2k+3) \\ &= \frac{k}{3}\left(4k^2+6k-1\right)+(2k+1)(2k+3) \\ &= \frac{1}{3}\left[4k^3+6k^2-k+3(2k+1)(2k+3)\right] \\ &= \frac{1}{3}\left[4k^3+18k^2+23k+9\right] \\ &= \frac{k+1}{3}\left(4k^2+14k+9\right) \qquad \ldots(\text{using (i)}) \\ &= \text{RHS}\end{aligned}$$

Question 14 (b) (i)

Consider $(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$, and let $x=1$ to get $2^n = \sum_{r=0}^{n} \binom{n}{r}$, as required.

Question 14 (b) (ii)

Consider $(1+x)^n = \sum_{r=0}^{n} \binom{n}{k} x^r$. Differentiate both sides with respect to *x*.

$$n(1+x)^{n-1} = \sum_{r=1}^{n} r\binom{n}{r} x^{r-1}$$

$$n2^{n-1} = \sum_{r=1}^{n} r\binom{n}{r} \qquad \ldots(\text{by letting } x=1)$$

Question 14 (b) (iii)

From part (i), we have

$$\begin{aligned} 2^n &= \sum_{r=0}^{n} \binom{n}{r} \\ &= \binom{n}{0} + \sum_{r=1}^{n} \binom{n}{r} \end{aligned}$$

Hence, $2^n - 1 = \sum_{r=1}^{n} \binom{n}{r}$.

So starting with the sum given in the left hand side, we have

$$\begin{aligned} \sum_{r=1}^{n} (2r-n)\binom{n}{r} &= \sum_{r=1}^{n} (2r)\binom{n}{r} - \sum_{r=1}^{n} n\binom{n}{r} \\ &= 2\underbrace{\sum_{r=1}^{n} r\binom{n}{r}}_{\text{part (ii)}} - n\underbrace{\sum_{r=1}^{n} \binom{n}{r}}_{\text{found earlier}} \\ &= 2\left(n2^{n-1}\right) - n\left(2^n - 1\right) \\ &= n \end{aligned}$$

Question 14 (c) (i)

The tangent has equation $y = tx - at^2$. Solve simultaneously with the directrix $y = -a$.

$$\begin{aligned} -a &= tx - at^2 \\ tx &= at^2 - a \\ x &= a\left(t - \frac{1}{t}\right) \end{aligned}$$

Hence, *D* has coordinates $\left(a\left(t - \frac{1}{t}\right), -a\right)$.

Question 14 (c) (ii)

Note that *R* has the same *x* coordinate as *D*, by construction. Hence, all that remains to find is the *y* coordinate of *R*, which can be found by substituting $x = a\left(t - \frac{1}{t}\right)$ into the equation of the normal at *T*, which is $x + ty = at^3 + 2at$.

$$\begin{aligned} a\left(t - \frac{1}{t}\right) + ty &= at^3 + 2at \\ ty &= at^3 + at + \frac{a}{t} \\ y &= a + a\left(t^2 + \frac{1}{t^2}\right) \end{aligned}$$

Hence, the coordinates of *R* are $\left(a\left(t - \frac{1}{t}\right), a + a\left(t^2 + \frac{1}{t^2}\right)\right)$.

Let $x = a\left(t - \frac{1}{t}\right)$, $y = a + a\left(t^2 + \frac{1}{t^2}\right)$. Eliminate the parameter *t* by first squaring the *x*.

$$\begin{aligned} x^2 &= a^2\left(t^2 + \frac{1}{t^2} - 2\right) \\ &= a^2\left(\frac{y-a}{a} - 2\right) \\ &= a(y - 3a) \end{aligned}$$

which is another parabola.

Question 14 (c) (iii)

Since we have the locus $x^2 = a(y-3a)$ in standard form, we may read off the focal length, which is a quarter of the coefficient of *y*. Hence the focal length is $a/4$.

Question 14 (c) (iv)

Recall that from the coordinates of *R*, we have $x = a\left(t - \frac{1}{t}\right),\ y = a + a\left(t^2 + \frac{1}{t^2}\right)$.

Hence

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ &= \frac{a\left(2t - \frac{2}{t^3}\right)}{a\left(1 + \frac{1}{t^2}\right)} \\ &= \frac{2\left(t^2 - 1\right)}{t}\end{aligned}$$

This must be parallel to the tangent at *T*, since normal is shared amongst both parabolas.

$$\begin{aligned}\frac{2\left(t^2 - 1\right)}{t} &= t \\ t &= \pm\sqrt{2}\end{aligned}$$