2013 NSW BOS Mathematics Extension 1 Solutions

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Section I

1	2	3	4	5	6	7	8	9	10
C	D	C	D	A	B	A	D	B	C

Q1 $P(x)=x^3-4x^2-6x+k$ and $x-2$ is a factor

$\therefore P(2)=2^3-4\times2^2-6\times2+k=0$, $\therefore k=20$ C

Q2 $y=f^{-1}(x)$ is the reflection of $y=f(x)$ in the line $y=x$. D

Q3 $\angle ABC=\frac{1}{2}\left(2\pi-\frac{3\pi}{5}\right)=\frac{7\pi}{10}$ C

Q4 $y=x(1-x)^3(3-x)^2=-x(x-1)^3(x-3)^2$

$(x-1)^3$ indicates a stationary point of inflection on the *x*-axis at $x=1$, and $(x-3)^2$ indicates a turning point on the *x*-axis at $x=3$. $y\to-\infty$ as $x\to+\infty$ due to the negative sign. D

Q5 $u=1+2x$, $x=\frac{1}{2}(u-1)$, $dx=\frac{1}{2}du$

$\int x\sqrt{1+2x}dx=\int\frac{1}{2}(u-1)\sqrt{u}\,\frac{1}{2}du=\frac{1}{4}\int(u-1)\sqrt{u}\,du$ A

Q6 $\sin 2x=a$, $2x=2k\pi+\sin^{-1}a$ or $(2k+1)\pi-\sin^{-1}a$

$\therefore 2x=n\pi+(-1)^n\sin^{-1}a$ which gives the former if *n* is even; and the latter if *n* is odd.

$\therefore x=\frac{n\pi+(-1)^n\sin^{-1}a}{2}$ B

Q7 The youngest can sit at any one of the 8 seats, $\therefore$ in 8 ways. The second youngest can sit on either side of the youngest, $\therefore$ in 2 ways. The remaining 6 of the family can sit around the table in 6! ways. $\therefore$ total number of ways is $8\times2\times6!$

Without restrictions the total number of ways is 8!

$\therefore \Pr(youngest2together)=\frac{8\times2\times6!}{8!}=\frac{2\times6!}{7!}$ A

Q8 $\sin\theta=\frac{5}{13}$ and $\frac{\pi}{2}<\theta<\pi$, $\therefore \cos\theta=-\sqrt{1-\sin^2\theta}=-\frac{12}{13}$

$\therefore \sin2\theta=2\sin\theta\cos\theta=2\times\frac{5}{13}\times\frac{-12}{13}=-\frac{120}{169}$ D

Q9 The given diagram shows the graph of $y=\sin^{-1}x$ translated upwards by $\frac{\pi}{2}$ units. B

Q10 $|x+2|+|x-3|=\begin{cases}-2x+1 & when \quad x<-2\\ 5 & when \quad -2\le x\le3\\ 2x-1 & when \quad x>3\end{cases}$

$x^2-x-6\le0$, $(x+2)(x-3)\le0$, $\therefore -2\le x\le3$ C

Section II

Q11a $2x^3-3x^2-11x+7=2(x-\alpha)(x-\beta)(x-\gamma)$, $\therefore \alpha\beta\gamma=-\frac{7}{2}$

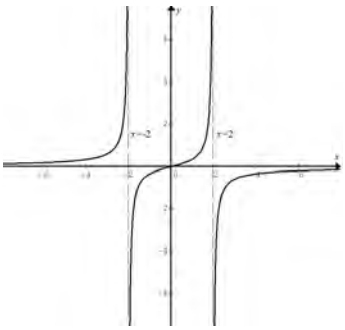
Q11b $\int\frac{1}{\sqrt{49-4x^2}}dx=\int\frac{1}{\sqrt{7^2-(2x)^2}}dx=\frac{1}{2}\sin^{-1}\left(\frac{2x}{7}\right)+c$

Q11c $n=10$, $p=\frac{1}{4}$, $\Pr(X=7)=\binom{10}{7}\left(\frac{1}{4}\right)^7\left(\frac{3}{4}\right)^3$

Q11di $f(x)=\frac{x}{4-x^2}$, $f'(x)=\frac{4-x^2-x(-2x)}{(4-x^2)^2}=\frac{4+x^2}{(4-x^2)^2}$

Since both $4+x^2$ and $(4-x^2)^2$ are positive, $\therefore f'(x)>0$ for all *x* in the domain of $f(x)$.

Q11dii $f(x)=\frac{x}{4-x^2}=-\frac{x}{(x+2)(x-2)}$



Q11e $\lim_{x\to0}\frac{\sin\frac{x}{2}}{3x}=\lim_{x\to0}\frac{\sin\frac{x}{2}}{6\times\frac{x}{2}}=\frac{1}{6}\times\lim_{x\to0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}=\frac{1}{6}\times1=\frac{1}{6}$

Q11f $u=e^{3x}$, $\frac{1}{3}du=e^{3x}dx$. When $x=0$, $u=1$; $x=\frac{1}{3}$, $u=e$.

$\int_0^{\frac{1}{3}}\frac{e^{3x}}{e^{6x}+1}dx=\frac{1}{3}\int_1^e\frac{du}{u^2+1}=\frac{1}{3}\left[\tan^{-1}u\right]_1^e=\frac{1}{3}\left(\tan^{-1}e-\frac{\pi}{4}\right)$

Q11g $\frac{d}{dx}\left(x^2\sin^{-1}5x\right)=\left(\sin^{-1}5x\right)(2x)+\left(x^2\right)\left(\frac{5}{\sqrt{1-(5x)^2}}\right)$

$=x\left(2\sin^{-1}5x+\frac{5x}{\sqrt{1-25x^2}}\right)$

Q12ai

$\sqrt{3}\cos x-\sin x=2\cos(x+\alpha)=2\cos x\cos\alpha-2\sin x\sin\alpha$

$\therefore \cos\alpha=\frac{\sqrt{3}}{2}$ and $\sin\alpha=\frac{1}{2}$, $\therefore \tan\alpha=\frac{1}{\sqrt{3}}$, $\alpha=\frac{\pi}{6}$

$\therefore \sqrt{3}\cos x-\sin x=2\cos\left(x+\frac{\pi}{6}\right)$

Q12aii $\sqrt{3}\cos x = 1+\sin x$ and $0 < x < 2\pi$

$\therefore \sqrt{3}\cos x - \sin x = 2\cos\left(x+\frac{\pi}{6}\right) = 1$, $\therefore \cos\left(x+\frac{\pi}{6}\right) = \frac{1}{2}$

$\therefore x+\frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}$, $\therefore x = \frac{\pi}{6}, \frac{3\pi}{2}$

Q12b $Volume = \int_0^{\frac{3\pi}{2}} \pi y^2 dx = \int_0^{\frac{3\pi}{2}} \pi\left(3\sin\frac{x}{2}\right)^2 dx = \frac{9\pi}{2}\int_0^{\frac{3\pi}{2}} (1-\cos x)dx$

$= \frac{9\pi}{2}\left[x-\sin x\right]_0^{\frac{3\pi}{2}} = \frac{9\pi}{2}\left(\frac{3\pi}{2}+1\right)$ cubic units

Q12c $T = A + Be^{-kt}$

When $t=0$, $T=80$; $t=10$, $T=60$; $t\to\infty$, $T\to 22$.

$\therefore A+B=80$, $A+Be^{-10k}=60$ and $A=22$

$\therefore B=58$ and $e^{-10k} = \frac{19}{29}$, $\therefore k = \frac{1}{-10}\log_e\frac{19}{29} \approx 0.04229$

$\therefore T \approx 22+58e^{-0.04229t}$, $\therefore t \approx \frac{1}{-0.04229}\log_e\frac{T-22}{58}$

When $T=40$, $t \approx 28$ minutes

Q12di $D(t) = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$ where (x_1, y_1) is (t, t^2+3) and $ax+by+c=0$ is $2x-y-1=0$.

$\therefore D(t) = \frac{|2t-t^2-3-1|}{\sqrt{2^2+(-1)^2}} = \frac{|2t-t^2-4|}{\sqrt{5}} = \frac{|t^2-2t+4|}{\sqrt{5}} = \frac{t^2-2t+4}{\sqrt{5}}$

since $t^2-2t+4 = (t-1)^2+3 > 0$ for all $t \in R$.

Q12dii Let $\frac{dD}{dt} = \frac{2t-2}{\sqrt{5}} = 0$, $\therefore t=1$ when P is closest to ℓ.

Q12diii $y = x^2+3$, when $t=1$, $P(1,4)$, $m = \frac{dy}{dx} = 2x = 2(1) = 2$

$\therefore$ the tangent at P is parallel to ℓ.

Q12e $v^2+9x^2 = k$, $\frac{1}{2}v^2 = \frac{k}{2}-\frac{9}{2}x^2$, $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -9x$

$\therefore$ the particle moves in simple harmonic motion, and the period is $\frac{2\pi}{\sqrt{9}} = \frac{2\pi}{3}$

Q13ai Given $V = \frac{4}{3}\pi r^3$, $A = 4\pi r^2$ and $\frac{dV}{dt} = -10^{-4}A$

$\therefore \frac{dV}{dr} = 4\pi r^2 = A$ and $\frac{dV}{dt} = \frac{dV}{dr}\times\frac{dr}{dt} = -10^{-4}A$

$\therefore A\times\frac{dr}{dt} = -10^{-4}A$, $\therefore \frac{dr}{dt} = -10^{-4}$ is constant

Q13aii When $t=0$, $V = \frac{4}{3}\pi r^3 = 10^{-6}\text{ m}^3$, $\therefore r \approx 0.0062$ m

Since $\frac{dr}{dt} = -10^{-4}$ is constant, $\therefore$ *time taken* $\approx \frac{0.0062}{10^{-4}} \approx 62$ s

Q13bi Let $G(x, y)$ divides NT externally in the ratio $2:1$.

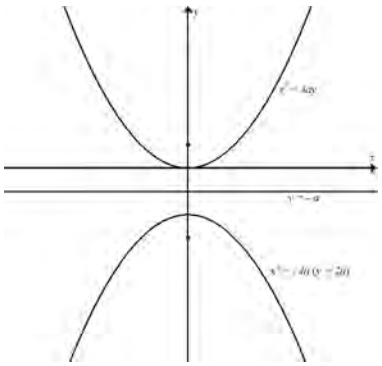
$\therefore T(ap, 0)$ is the midpoint of NG

$\therefore (ap, 0) = \left(\frac{0+x}{2}, \frac{2a+ap^2+y}{2}\right)$

$\therefore x = 2ap$ and $y = -2a-ap^2$, i.e. $G(2ap, -2a-ap^2)$

Q13bii $x = 2ap$, $p = \frac{x}{2a}$, $\therefore y = -2a-a\left(\frac{x}{2a}\right)^2$

$\therefore x^2 = -4a(y+2a)$. It is a parabola with directrix $y = a-2a = -a$ and focal length a, same directrix and focal length as the original parabola.



Q13ci Given $y = Vt\sin\theta - \frac{g}{2}t^2$, for projectile fired at A,

$y = ut\sin\alpha - \frac{g}{2}t^2$, $\frac{dy}{dt} = u\sin\alpha - gt$.

At maximum height, $\frac{dy}{dt} = u\sin\alpha - gt = 0$, $\therefore t = \frac{u\sin\alpha}{g}$

A13cii

For projectile fired at B, $y = wt\sin\beta - \frac{g}{2}t^2$, $\frac{dy}{dt} = w\sin\beta - gt$.

At maximum height, $\frac{dy}{dt} = w\sin\beta - gt = 0$, $\therefore t = \frac{w\sin\beta}{g}$.

Since they collide, $\therefore$ they reach the same maximum height at the same time, $\therefore \frac{u\sin\alpha}{g} = \frac{w\sin\beta}{g}$, $\therefore u\sin\alpha = w\sin\beta$

Q13ciii Given $x = Vt\cos\theta$, for projectile fired at A,

$x = u\left(\frac{u\sin\alpha}{g}\right)\cos\alpha = \frac{1}{g}(u\sin\alpha)(u\cos\alpha)$.

Similarly, for projectile fired at B, $x = \frac{1}{g}(w\sin\beta)(w\cos\beta)$.

$\therefore d = \frac{1}{g}(w\sin\beta)(w\cos\beta) + \frac{1}{g}(u\sin\alpha)(u\cos\alpha)$

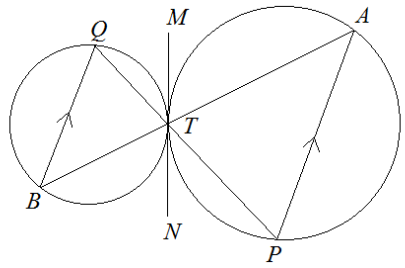
$= \frac{1}{g}((w\sin\beta)(w\cos\beta) + (u\sin\alpha)(u\cos\alpha))$

$= \frac{1}{g}((u\sin\alpha)(w\cos\beta) + (w\sin\beta)(u\cos\alpha))$

$= \frac{uw}{g}\sin(\alpha+\beta)$

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Q13d Draw common tangent MN, chords QT and PT.



$\angle QTM = \angle QBT$ and $\angle PTN = \angle PAT$ (angles in alternate segments)

$\angle QBT = \angle PAT$ (alternate angles)

$\therefore\ \angle QTM = \angle PTN$, $\therefore$ they are vertically opposite angles and $\therefore$ QTP is a straight line segment, i.e. Q, T and P are collinear.

Q14ai

$$\frac{1}{(k+1)^2} - \frac{1}{k} + \frac{1}{k+1} = \frac{k-(k+1)^2+k(k+1)}{k(k+1)^2} = \frac{-1}{k(k+1)^2} < 0$$

given $k > 0$.

Q14aii For $n = 2$, $\frac{1}{1^2} + \frac{1}{2^2} < 2 - \frac{1}{2}$ is true.

Assume the statement is true for $n = k$,

i.e. $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{k^2} < 2 - \frac{1}{k}$

then

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$$

since $\frac{1}{(k+1)^2} - \frac{1}{k} < -\frac{1}{k+1}$ from Q14ai.

$\therefore$ the statement is true for $n = k+1$

$\therefore$ it is true for all integers $n \geq 2$.

Q14bi The term consisting of x^{2n} in the expansion of $(1+x)^{4n}$ is $\binom{4n}{2n}\left(1^{2n}\right)\left(x^{2n}\right) = \binom{4n}{2n}x^{2n}$, $\therefore$ *coefficient* $= \binom{4n}{2n}$

Q14bii

$$\left(1+\left(x^2+2x\right)\right)^{2n} = \binom{2n}{0}1^0\left(x^2+2x\right)^{2n} + \binom{2n}{1}1^1\left(x^2+2x\right)^{2n-1}$$

$$+\binom{2n}{2}1^2\left(x^2+2x\right)^{2n-2} + ... + \binom{2n}{2n}1^{2n}\left(x^2+2x\right)^0$$

$$= \sum_{k=0}^{2n}\binom{2n}{k}\left(x^2+2x\right)^{2n-k} = \sum_{k=0}^{2n}\binom{2n}{k}\left(x(x+2)\right)^{2n-k}$$

$$= \sum_{k=0}^{2n}\binom{2n}{k}x^{2n-k}(x+2)^{2n-k}$$

Q14biii $\left(1+x^2+2x\right)^{2n} = \left((1+x)^2\right)^{2n} = (1+x)^{4n}$

$\therefore$ coefficient of x^{2n} in the expansion of $\left(1+x^2+2x\right)^{2n}$ is $\binom{4n}{2n}$.

$$\left(1+x^2+2x\right)^{2n} = \sum_{k=0}^{2n}\binom{2n}{k}x^{2n-k}(x+2)^{2n-k}$$

$$= \sum_{k=0}^{2n}\binom{2n}{k}\left[\binom{2n-k}{0}2^{2n-k}x^{2n-k} + \binom{2n-k}{1}2^{2n-k-1}x^{2n-k+1} + ...\right.$$

$$\left. .....+\binom{2n-k}{2n-k}2^0 x^{4n-2k}\right]$$

$$= \sum_{k=0}^{2n}\left[\binom{2n}{k}\binom{2n-k}{0}2^{2n-k}x^{2n-k} + \binom{2n}{k}\binom{2n-k}{1}2^{2n-k-1}x^{2n-k+1}\right.$$

$$\left. + ... + \binom{2n}{k}\binom{2n-k}{2n-k}2^0 x^{4n-2k}\right]$$

Collecting terms of x^{2n}:

$$\binom{2n}{0}\binom{2n-0}{0}2^{2n-0}x^{2n} + \binom{2n}{1}\binom{2n-1}{1}2^{2n-2}x^{2n} + ...$$

$$.....+\binom{2n}{n}\binom{2n-n}{2n-n}2^0 x^{2n}$$

$\therefore$ coefficient of x^{2n} in the expansion of $\left(1+x^2+2x\right)^{2n}$ is

$$2^{2n-0}\binom{2n}{0}\binom{2n-0}{0} + 2^{2n-2}\binom{2n}{1}\binom{2n-1}{1} + ... + 2^0\binom{2n}{n}\binom{2n-n}{2n-n}$$

$$= \sum_{k=0}^{n}2^{2n-2k}\binom{2n}{k}\binom{2n-k}{k}$$

$$\therefore \binom{4n}{2n} = \sum_{k=0}^{n}2^{2n-2k}\binom{2n}{k}\binom{2n-k}{k}$$

Q14ci Given $t_0 = 0.5$ is an approximate root of $e^t - \frac{1}{t} = 0$.

Let $f(t) = e^t - \frac{1}{t}$, $f'(t) = e^t + \frac{1}{t^2}$

$$t_1 = t_0 - \frac{f(t_0)}{f'(t_0)} = 0.5 - \frac{e^{0.5} - \frac{1}{0.5}}{e^{0.5} + \frac{1}{0.5^2}} \approx 0.56$$

Q14cii Let $t = rx$, $\therefore$ the two curves are $y = e^t$ and $y = \log_e\left(\frac{t}{r}\right)$,

and the derivatives are $\frac{dy}{dt} = e^t$ and $\frac{dy}{dt} = \frac{1}{t}$ respectively.

At their point of intersection, the two curves have a common tangent.

$\therefore\ e^t = \frac{1}{t}$, $\therefore\ t \approx 0.56$ from Q14cii.

Also, $\log_e\left(\frac{t}{r}\right) = e^t$ $\therefore\ \log_e\left(\frac{0.56}{r}\right) \approx e^{0.56}$, $\frac{0.56}{r} \approx e^{e^{0.56}}$, $\therefore\ r \approx 0.097$

Please inform mathline@itute.com re conceptual and/or mathematical errors.

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