2013 HSC Examination

Mathematics Extension 1

SOLUTIONS

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Multiple Choice

1. C
2. D
3. C
4. D
5. A
6. B
7. A
8. D
9. B
10. C

Brief Explanations

Question 1 By the Factor Theorem, $P(2)=0$.

Question 2 Draw $y=x$ and the answer is clearly (D) by symmetry about $y=x$.

Question 3 The reflex $\angle AOC$ is double $\angle ABC$.

Question 4 Re-arrange the polynomial to get $P(x)=-x(x-1)^3(x-3)^2$.

Question 5 Standard substitution problem.

Question 6 Standard general solution problem.

Question 7 Group the two and arrange in a circle, so 6!, but they swap, so 2!.

Question 8 Construct a 5-12-13 right angled triangle and use $\sin 2\theta = 2\sin\theta\cos\theta$.

Question 9 Standard inverse trigonometry sketching.

Question 10 Work out the cases being $x<-2$, $-2\le x\le 3$ and $x>3$ to find that the required result is $-2\le x\le 3$. Only (C) satisfies this.

Written Response

Question 11 (a)

Standard product of roots, so using that we have $-\frac{7}{2}$

Question 11 (b)

$$\frac{1}{2}\sin^{-1}\left(\frac{2x}{7}\right)+C$$

Question 11 (c)

Binomial probability scenario, with $P(\text{correct})=\frac{1}{4}$ and $P(\text{incorrect})=\frac{3}{4}$.

$$P(X=7)=\binom{10}{7}\left(\frac{1}{4}\right)^7\left(\frac{3}{4}\right)^3$$

Question 11 (d) (i)

We differentiate the function first.

$$\begin{aligned} f'(x) &= \frac{4-x^2+2x^2}{\left(4-x^2\right)^2} \\ &= \frac{4+x^2}{\left(4-x^2\right)^2} \end{aligned}$$

Since all terms are squared and $f'(x)\neq 0$, we have $f'(x)>0$.

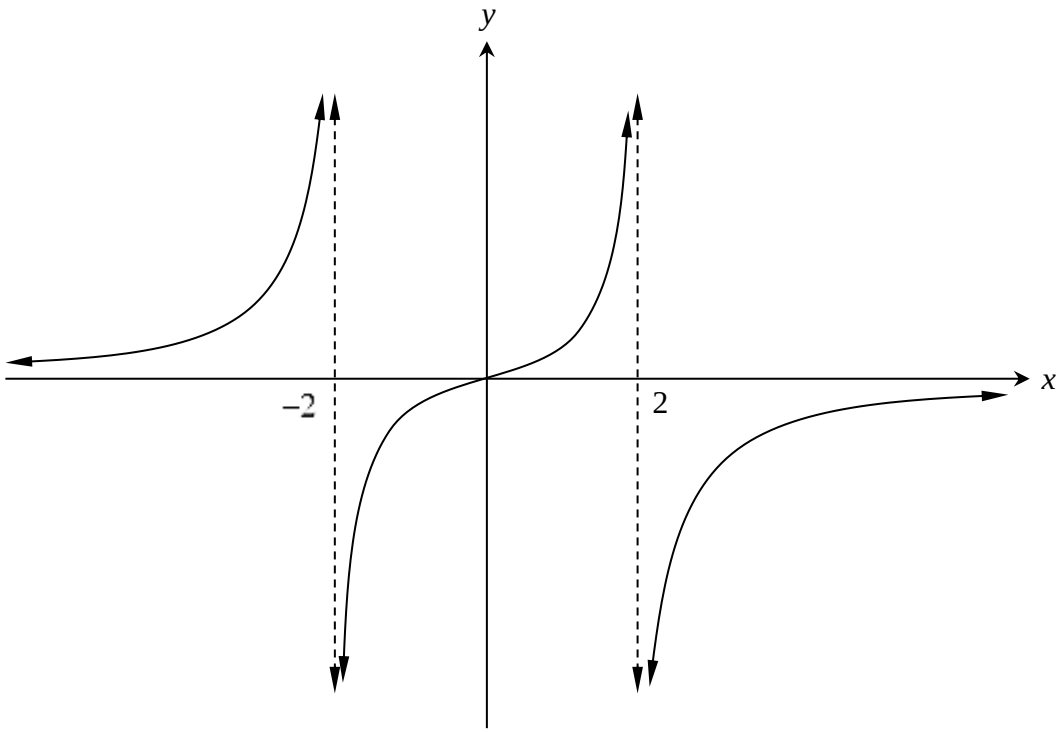
Question 11 (d) (ii)

Clearly, there are vertical asymptotes at $x = \pm 2$.

As $x \to \infty$ and $x \to -\infty$, we have $y \to 0^-$ and $y \to 0^+$ respectively.

Also note that the function is odd and so it has its *x* and *y* intercept at the origin.

Putting the pieces together, along with the fact that it is always increasing, we have



Question 11 (e)

We make use of the familiar limit $\lim\limits_{x \to 0} \dfrac{\sin\theta}{\theta} = 1$.

$$\lim_{x \to 0} \frac{\sin\left(\frac{x}{2}\right)}{3x} = \lim_{x \to 0} \frac{\sin\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)} \times \frac{1}{6}$$

$$= \frac{1}{6}$$

Question 11 (f)

$$u = e^{3x} \Rightarrow du = 3e^{3x}dx \Rightarrow \frac{1}{3}du = e^{3x}dx$$

$$\begin{aligned} x &= \frac{1}{3} \quad \Rightarrow \quad u = e \\ x &= 0 \quad \Rightarrow \quad u = 1 \end{aligned}$$

$$\begin{aligned} I &= \frac{1}{3}\int_1^e \frac{du}{u^2+1} \\ &= \frac{1}{3}\tan^{-1}u\Big|_1^e \\ &= \frac{1}{3}\left(\tan^{-1}e - \tan^{-1}1\right) \\ &= \frac{1}{3}\left(\tan^{-1}e - \frac{\pi}{4}\right) \end{aligned}$$

Question 11 (g)

Using the Product Rule, we have

$$\frac{d}{dx}\left(x^2\sin^{-1}5x\right) = 2x\sin^{-1}5x + \frac{5x^2}{\sqrt{1-25x^2}}$$

Question 12 (a) (i)

Comparing the two expressions, we have $2\cos x\cos\alpha - 2\sin x\sin\alpha \equiv \sqrt{3}\cos x - \sin x$.

Hence, by equating say $\cos x$ and noting that α is acute, we have

$$\begin{aligned} 2\cos\alpha &= \sqrt{3} \\ \cos\alpha &= \frac{\sqrt{3}}{2} \\ \alpha &= \frac{\pi}{6} \end{aligned}$$

Hence, the required expression is $2\cos\left(x + \frac{\pi}{6}\right)$.

Question 12 (a) (ii)

$$\sqrt{3}\cos x - \sin x = 1$$
$$2\cos\left(x + \frac{\pi}{6}\right) = 1$$
$$\cos\left(x + \frac{\pi}{6}\right) = \frac{1}{2}$$
$$x + \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}$$
$$x = \frac{\pi}{6}, \frac{3\pi}{2}$$

Question 12 (b)

We do the usual squaring the function and placing a π in front. As to be expected, we must use the formula $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ to help simplify our integrand.

$$V = \pi\int_0^{\frac{3\pi}{2}} 9\sin^2\left(\frac{x}{2}\right)dx$$
$$= \frac{9\pi}{2}\int_0^{\frac{3\pi}{2}} 1 - \cos x\,dx$$
$$= \frac{9\pi}{2}\Big[\ x - \sin x\ \Big]_0^{\frac{3\pi}{2}}$$
$$= \frac{9\pi}{2}\left[\frac{3\pi}{2} + 1\right]$$
$$= \frac{9\pi}{4}(3\pi + 2)$$

Question 12 (c)

As $t \to \infty, T \to A$, so A must be the room temperature, which is $22^\circ C$.

When $t = 0, T = 80$ and when $t = 10, T = 60$.

Subbing in $t = 0, T = 80$, we have

$$80 = 22 + Be^0$$
$$B = 58$$

Subbing in $t = 10, T = 60$, we have

$$60 = 22 + 58e^{-10k}$$
$$38 = 58e^{-10k}$$
$$e^{-10k} = \frac{38}{58}$$
$$-10k = \ln\left(\frac{19}{29}\right)$$
$$k = -\frac{1}{10}\ln\left(\frac{19}{29}\right)$$
$$= \frac{1}{10}\ln\left(\frac{29}{19}\right)$$

Now that we have everything, we can work out the time it takes to drop down to $40^\circ C$.

$$40 = 22 + 58e^{-kt} \qquad \ldots\left(\text{where } k = \frac{1}{10}\ln\left(\frac{29}{19}\right)\right)$$
$$18 = 58e^{-kt}$$
$$e^{-kt} = \frac{9}{29}$$
$$-kt = \ln\left(\frac{9}{29}\right)$$
$$t = -\frac{1}{k}\ln\left(\frac{9}{29}\right)$$
$$= -10 \times \frac{\ln\left(\frac{9}{29}\right)}{\ln\left(\frac{29}{19}\right)}$$
$$\approx 28 \text{ minutes}$$

Question 12 (d) (i)

Using the Perpendicular Distance Formula using $2x-y-1=0$ and $P\left(t,t^2+3\right)$ we have

$$\begin{aligned} D(t) &= \left|\frac{2t-\left(t^2+3\right)-1}{\sqrt{5}}\right| \\ &= \left|\frac{2t-t^2-4}{\sqrt{5}}\right| \\ &= \frac{t^2-2t+4}{\sqrt{5}} \end{aligned}$$

Note that the quadratic is positive definite so we can omit the absolute value symbols.

Question 12 (d) (ii)

When *P* is closest to *l*, $D(t)$ is minimised, so we find the minimum value using the familiar expression $t=-\frac{b}{2a}$, which yields $t=\frac{2/\sqrt{5}}{2/\sqrt{5}}=1$.

Question 12 (d) (iii)

We now find the gradient of the tangent when $t=1$.

Using Parametric Differentiation, we have $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=2t$ so the gradient is 2 and thus, the tangent is parallel to *l*.

Question 12 (e)

We make use of the familiar formula $\ddot{x}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)=-n^2x$.

$$\frac{1}{2}v^2=\frac{1}{2}\left(k-9x^2\right)$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right)=-9x$$

Hence we can see that $n=3$, from which we deduce that the period is $T=\frac{2\pi}{n}=\frac{2\pi}{3}$.

Question 13 (a) (i)

We use the Chain Rule to get

$$\begin{aligned}\frac{dr}{dt} &= \frac{dr}{dV}\times\left(-10^{-4}A\right)\\ &= \frac{1}{dV/dr}\left(-10^{-4}A\right)\\ &= \frac{1}{4\pi^2}\left(-10^{-4}A\right)\\ &= \frac{1}{A}\left(-10^{-4}A\right)\\ &= -10^{-4}\end{aligned}$$

And hence it is constant.

Question 13 (a) (ii)

Since the radius reduces at a constant rate of 10^{-4} metres per second, all that remains is to work out the radius of the raindrop, then to find how long it takes for radius to be zero.

$$\begin{aligned}\frac{4}{3}\pi r^3 &= 10^{-6}\\ r^3 &= \frac{3}{4\pi}10^{-6}\\ r &= 10^{-2}\sqrt[3]{\frac{3}{4\pi}}\end{aligned}$$

So to calculate the time, we place that as a ratio over the rate.

$$\begin{aligned}T &= \frac{10^{-2}\sqrt[3]{\frac{3}{4\pi}}}{10^{-4}}\\ &= 10^{2}\sqrt[3]{\frac{3}{4\pi}}\\ &\approx 62 \text{ seconds}\end{aligned}$$

Question 13 (b) (i)

Since the division is external, let the ratio be $2:-1$, so

$$G\left(\frac{2(ap)-1(0)}{1},\frac{2(0)-\left(2a+ap^2\right)}{1}\right)=G\left(2ap,-2ap-ap^2\right)$$

Question 13 (b) (ii)

We are now essentially finding the locus of *G*.

$$x=2ap \Rightarrow p=\frac{x}{2a}$$

$$\begin{aligned} y &= -2a-ap^2 \\ &= -2a-a\left(\frac{x}{2a}\right)^2 \\ &= -2a-\frac{x^2}{4a} \\ 4ay &= -8a^2-x^2 \\ x^2 &= -8a^2-4ay \\ -x^2 &= 4a\left(2a+y\right) \end{aligned}$$

And we can see that the focal length is *a* and its vertex is $(0,-2a)$.

Since the vertex $(0,-2a)$, and it is an upside-down parabola, the directrix must be $y=-a$, which is the same as the directrix of the original parabola.

Question 13 (c) (i)

For the projectile from *A* to reach maximum height, we let $\dot{y}=0$.

$$\begin{aligned} u\sin\alpha-gt &= 0 \\ gt &= u\sin\alpha \\ t &= \frac{u}{g}\sin\alpha \end{aligned}$$

Question 13 (c) (ii)

Since both particles collide when they reach maximum height, we can equate the times that they take to reach maximum height.

From (i), we have $t_A = \frac{u}{g}\sin\alpha$ and similarly, we have $t_B = \frac{w}{g}\sin\beta$, so

$$\frac{u}{g}\sin\alpha = \frac{w}{g}\sin\beta$$
$$u\sin\alpha = w\sin\beta$$

Question 13 (c) (iii)

Since they also collide, we can equate their respective x and y coordinates.

For particle from A:

$x = ut\cos\alpha$ and $y = ut\sin\alpha - \frac{g}{2}t^2$

For particle from B:

$x = d - wt\cos\beta$, taking the origin to be A, and $y = wt\sin\beta - \frac{g}{2}t^2$

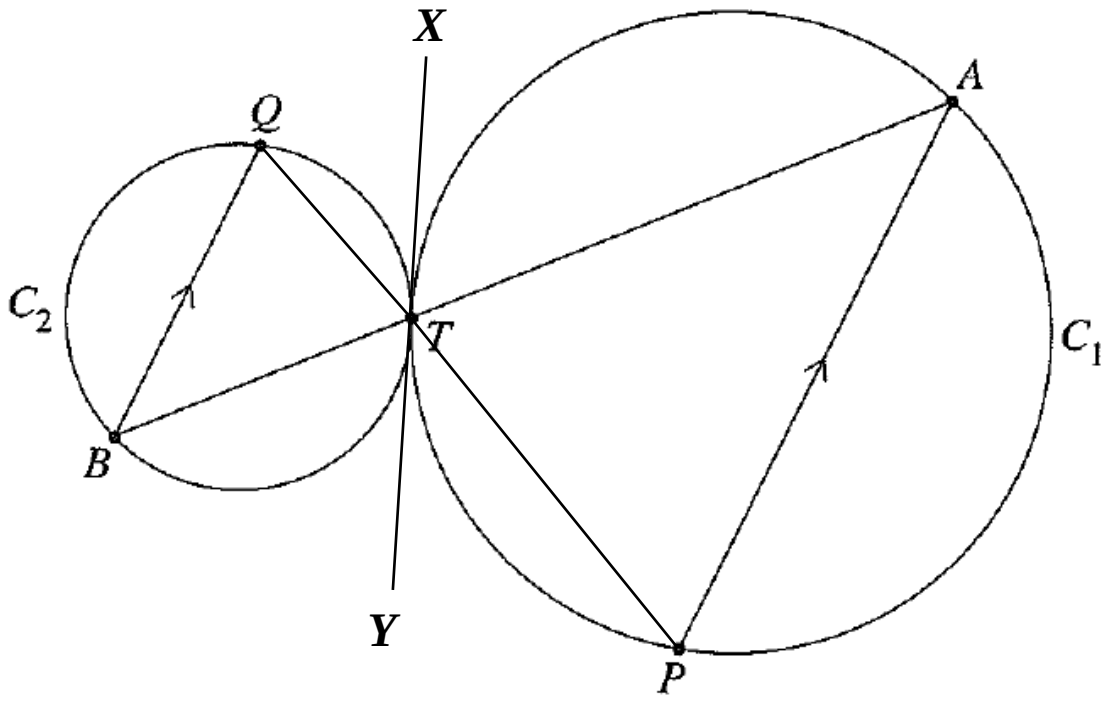
Equating the x coordinates, we have

$$d - wt\cos\beta = ut\cos\alpha$$
$$d = ut\cos\alpha + wt\cos\beta$$
$$= u\left(\frac{w}{g}\sin\beta\right)\cos\alpha + w\left(\frac{u}{g}\sin\alpha\right)\cos\beta$$
$$= \frac{uw}{g}\sin(\alpha+\beta)$$

We could make the above substitutions since the times are equal, so it does not matter which one we substitute into which expression.

Question 13 (d)

We construct a common tangent at T and label the tangent XY.



$$\begin{aligned}
\angle XTQ &= \angle QBT && \ldots(\text{Alternate Segment Theorem}) \\
&= \angle TAP && \ldots(\text{Alternate angles, } QB//PA) \\
&= \angle YTP && \ldots(\text{Alternate Segment Theorem})
\end{aligned}$$

Hence, vertically opposite angles are equal and since XY is a straight line, PTQ must also form a straight line, and the points are thus collinear.

Question 14 (a) (i)

$$\begin{aligned}\text{LHS} &= \frac{k-(k+1)^2+k(k+1)}{k(k+1)^2} \\ &= \frac{k-k^2-2k-1+k^2+k}{k(k+1)^2} \\ &= \frac{-1}{k(k+1)^2} \\ &< 0 \qquad \ldots(\text{since } k>0)\end{aligned}$$

Question 14 (a) (ii)

Base Case: $n=2$

$$\text{LHS} = \frac{1}{1^2}+\frac{1}{2^2} = \frac{5}{4} \qquad \text{RHS} = 2-\frac{1}{2} = \frac{3}{2}$$

$\therefore$ LHS < RHS

Inductive Hypothesis: $n=k$, where $k \geq 2$.

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{k^2} < 2-\frac{1}{k}$$

Inductive Step: $k \Rightarrow k+1$.

Required to prove: $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{k^2}+\frac{1}{(k+1)^2} < 2-\frac{1}{k+1}$

$$\begin{aligned}\text{LHS} &< 2-\frac{1}{k}+\frac{1}{(k+1)^2} && \ldots(\text{by the Induction Hypothesis}) \\ &< 2-\frac{1}{(k+1)^2} && \ldots(\text{from part (i)}) \\ &< 2-\frac{1}{k+1} && \ldots(\text{since a larger denominator means a smaller overall value}) \\ &= \text{RHS}\end{aligned}$$

Hence, proven by Induction.

Question 14 (b) (i)

$$\binom{4n}{2n}$$

Question 14 (b) (ii)

We group terms appropriately, noting that the RHS has $(x+2)$, which implies we must take out *x*.

$$\begin{aligned}\left(1+x^2+2x\right)^{2n} &= \left(1+x(x+2)\right)^{2n} \\ &= \sum_{k=0}^{n}\binom{n}{k}\left[x(x+2)\right]^{2n-k} \\ &= \sum_{k=0}^{n}\binom{n}{k}x^{2n-k}(x+2)^{2n-k}\end{aligned}$$

Question 14 (b) (iii)

First of all, we note that the LHS of (ii) is a perfect square, so it can be expressed as $(x+1)^{4n}$.

We must equate the coefficients of x^{2n}, since the LHS is the coefficient of x^{2n} from (i).

Firstly, note that there are many ways to acquire x^{2n}. From the given expression, we can acquire such a value by letting $k = 0, 1, 2, \ldots, n$.

Also, each term is actually multiplied by $\binom{2n}{k}$ because of the expression in (ii).

The provided expansion is just for the terms in *x*.

$$k=0 \quad \Rightarrow \quad \binom{2n}{0}\binom{2n}{0}2^{2n}x^{2n}$$

$$k=1 \quad \Rightarrow \quad \binom{2n}{1}\binom{2n-1}{1}2^{2n-2}x^{2n}$$

$$k=2 \quad \Rightarrow \quad \binom{2n}{2}\binom{2n-2}{2}2^{2n-4}x^{2n}$$

…

$$k=n \quad \Rightarrow \quad \binom{2n}{n}\binom{n}{n}2^{0}x^{2n}$$

From (i), the coefficient of $(1+x)^{4n}$ is $\binom{4n}{2n}$ and thus we can equate the two coefficients.

Doing so, and using sigma notation to express our result, we have the required expression.

$$\binom{4n}{2n} = \sum_{k=0}^{n} 2^{2n-2k} \binom{2n}{k} \binom{2n-k}{k}.$$

Question 14 (c) (i)

Set $f(t) = e^t - \frac{1}{t}$, so $f'(t) = e^t + \frac{1}{t^2}$.

$$\begin{aligned} t_1 &= t_0 - \frac{f(t_0)}{f'(t_0)} \\ &= 0.5 - \frac{e^{0.5} - \frac{1}{0.5}}{e^{0.5} + \frac{1}{0.5^2}} \\ &\approx 0.56 \end{aligned}$$

Question 14 (c) (ii)

Let the two curves intersect at say $x = k$.

Since they must share the same *y* coordinate, we have

$$e^{kr} = \ln k$$

Since the two curves are tangential, the gradients of their tangents must be equal at their intersection.

$$re^{rk} = \frac{1}{k}$$

$$e^{kr} = \frac{1}{kr}$$

But note that this looks exactly the same as the expression in (i), so make the substitution $kr = t$.

Doing so, we know that $t_1 \approx 0.56$ and so $kr = t_1 \approx 0.56$.

But since $e^{kr} = \ln k$, we have $e^{t_1} = \ln k$ and so $k \approx e^{e^{t_1}} = 5.758$. Hence, $r = \frac{t_1}{k} \approx 0.097$.