2002 HSC Full Solutions

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QUESTION 1

(a)

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{3\sin 3x}{3x}$$
$$= 3\lim_{x\to 0}\frac{\sin 3x}{3x}$$
$$= 3$$

(b)

$$\int_0^{\frac{\pi}{6}} \sec 2x \tan 2x dx = \frac{1}{2}\left[\sec 2x\right]_0^{\frac{\pi}{6}} \qquad \text{(see standard integrals)}$$
$$= \frac{1}{2}\left(\sec\frac{\pi}{3} - \sec 0\right)$$
$$= \frac{1}{2}(2-1) = \frac{1}{2}$$

(c)

$$\frac{d}{dx}3x^2 \ln x = 3x^2.\frac{d}{dx}(\ln x) + \ln x.\frac{d}{dx}(3x^2)$$
$$= 3x^2.\frac{1}{x} + \ln x.6x$$
$$= 3x + 6x.\ln x \quad for \quad x > 0$$

(d)

The domain of $y = \sin^{-1} x$ *is* $-1 \le x \le 1$

$\therefore -1 \le \frac{x}{2} \le 1 \quad i.e. -2 \le x \le 2$

The range of $y = \sin^{-1} x$ *is* $-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$

$\therefore y = 3\sin^{-1}\frac{x}{2}$ *is* $-\frac{3\pi}{2} \le 3\sin^{-1}\frac{x}{2} \le \frac{3\pi}{2}$

(e)

x = 3t and y = $2t^2$ $\therefore t = \frac{x}{3}$ *so* $y = 2\left(\frac{x}{3}\right)^2 \therefore y = \frac{2x^2}{9}$

(f)

$u = 1 - x^2 \therefore \frac{du}{dx} = -2x \therefore du = -2xdx$

At $x = 2$, $u = -3$ *At* $x = 3$, $u = -8$

$$\int_2^3 \frac{2x}{\left(1-x^2\right)^2} dx = \int_{-3}^{-8} \frac{-du}{u^2}$$
$$= -\int_{-3}^{-8} u^{-2} du = -\left[\frac{u^{-1}}{-1}\right]_{-3}^{-8}$$
$$= \left[\frac{1}{u}\right]_{-3}^{-8} = -\frac{1}{8} - \left(-\frac{1}{3}\right) = \frac{5}{24}$$

Question 2

(a)

$2^x = 3$

$\therefore \log 2^x = \log 3$

$x \log 2 = \log 3$

$x = \dfrac{\log 3}{\log 2} \approx 1.58$

(b)

$2\cos x = \sqrt{3}$

$\therefore \cos x = \dfrac{\sqrt{3}}{2}$

$\left(x = \dfrac{\pi}{6} \right)$

$x = 2n\pi \pm \dfrac{\pi}{6}$ n is an integer

(c)

$x^3 - 2x^2 + a \equiv (x+2)Q(x) + 3$

let $x = -2$

$-8 - 8 + a = (-2+2)Q(-2) + 3$

$-16 + a = 0.Q(-2) + 3$

$a = 19$

(d)

use $\sin^2 \alpha = \dfrac{1}{2}(1 - \cos 2\alpha)$

$2\int_0^{\frac{\pi}{4}} \sin^2 4x dx$

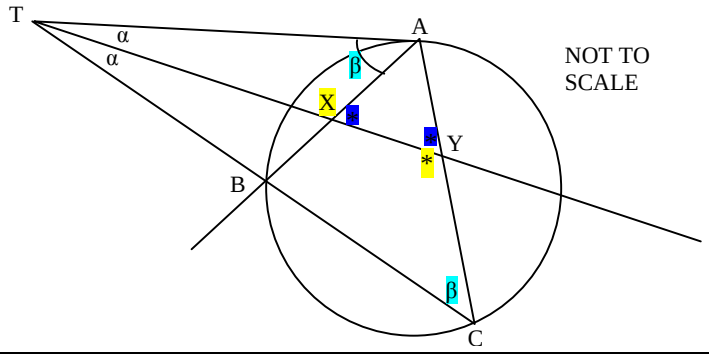
$= 2\int_0^{\frac{\pi}{4}} \dfrac{1}{2}(1 - \cos 8x)\, dx$

$= \int_0^{\frac{\pi}{4}} (1 - \cos 8x)\, dx$

$= \left[x - \dfrac{\sin 8x}{8} \right]_0^{\frac{\pi}{4}}$

$= \left(\dfrac{\pi}{4} - \dfrac{\sin 2\pi}{8} \right) - (0 - 0)$

$= \dfrac{\pi}{4} - \dfrac{0}{8} = \dfrac{\pi}{4}$

(e)	
(e)(i)	$\angle$ACB=β because the angle between a tangent and a chord equals the angle in the alternate segement
(e)(ii)	In ΔATX and ΔCTY: $\angle ATX = \angle CTY$ $(given)$ $\angle XAT = \angle YCT$ $(\angle ACB)$ $from\ part\,(i)$ $\therefore \Delta ATX \,|||\, \Delta CTY$ (AA) $\therefore \angle TXA = \angle TYC$ $corres.\ \angle's\ in\ similar\ \Delta's$ $Now\ \angle AXY = 180^0 - \angle TXA$ & $\angle AYX = 180^0 - \angle TYC$ $\therefore \angle AXY = \angle AYX$ (since $\angle TXA = \angle TYC$) $\therefore \Delta AXY$ is isosceles.

Question 3

(a) (i)

Number of circular permutations of 7 objects
$= (7 - 1)!$
$= 6! = 720$

(a) (ii)

Consider case where Kevin and Jill do sit together $5! \times 2$
Cases where they don't sit together $6! - 5! \times 2 = 480$

(b) (i)

$f(x) = e^x - 3x^2$

$f(3.7) \approx 40.4 - 42.1 < 0$ below axis

$f(3.8) \approx 44.7 - 43.2 > 0$ above axis

Since $f(x)$ continuous then a root exists

(b) (ii)

$f'(x) = e^x - 6x$

$x_1 = 3.8$

$$\therefore x_2 = x_1 - \frac{f(x)}{f'(x)}$$

$$= 3.8 - \frac{f(3.8)}{f'(3.8)}$$

≈ 3.74

(c) (i)

$T = 22 + Ae^{-kt}$

$$\frac{dT}{dt} = -kAe^{-kt}$$

$$\frac{dT}{dt} = -k(T - 22)$$

(c) (ii)

$T = 22 + Ae^{-kt}$

At $t = 0$ $T = 80^0$

$\therefore 80 = 22 + Ae^0$

$A = 80 - 22 = 58$

At $t = 10$ $T = 60$

$\therefore 60 = 22 + 58e^{-10k}$

$60 - 22 = 58e^{-10k}$

$38 = 58e^{-10k}$

$$e^{-10k} = \frac{38}{58}$$

$$\ln e^{-10k} = \ln \frac{38}{58}$$

$$-10k = \ln \frac{38}{58}$$

$$k = \frac{\ln \frac{38}{58}}{-10} \approx 0.0423$$

(c) (iii)	$T = 22 + 58e^{-0.0423t}$ At T=30^0 $30 = 22 + 58e^{-0.0423t}$ $8 = 58e^{-0.0423t}$ $\frac{8}{58} = e^{-0.0423t}$ $\ln\frac{8}{58} = -0.0423t$ $t = \frac{\ln\frac{8}{58}}{-0.0423} \approx 46.8484 \approx 47 \text{ minutes}$

Question 4

(a) (i)

Let $p = P(hit) \therefore p = \frac{2}{3}$

Let $q = P(miss) \therefore q = \frac{1}{3}$

$\therefore$ Probability of 9 hits in 10 trials

$= {}^{10}C_9 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^9 = 10 \times \frac{2^9}{3^{10}}$

(a) (ii)

Probability of 10 hits in 10 trials

$= {}^{10}C_{10} \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{10} = \frac{2^{10}}{3^{10}}$

Probability of 9 or 10 hits

$10 \times \frac{2^9}{3^{10}} + \frac{2^{10}}{3^{10}} = \frac{2^9}{3^{10}}(10+2) = 12 \times \frac{2^9}{3^{10}}$

$= 3 \times 4 \times \frac{2^9}{3^{10}} = \frac{2^{11}}{3^9}$

$\therefore$ Probability of fewer than 9 hits

$= 1 - \frac{2^{11}}{3^9}$

(b) (i)

$\alpha + \beta + \chi = -\frac{b}{a} = 2$

(b) (ii)

$\alpha\beta\chi = -\frac{d}{a} = -24$

(b) (iii)

Let roots be $-\alpha, \alpha$ & β

$-\alpha + \alpha + \beta = 2$

$\therefore \beta = 2$

Now $k = -\alpha\alpha + (-\alpha)\beta + \alpha\beta$

$= -\alpha^2$

And $-\alpha\alpha\beta = -24$

$-\alpha^2\beta = -24$

$-\alpha^2 2 = -24$

$\alpha^2 = 12$

$\therefore k = -12$ since $k = -\alpha^2$

(c)(i)	$\ddot{x} = -16x$ $\therefore \frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right) = -16x$ $\therefore \frac{1}{2}\dot{x}^2 = \int -16x dx$ $\therefore \frac{1}{2}\dot{x}^2 = -8x^2 + C$ At $x = 1\ \dot{x} = 4\ \therefore C = 16$ $\therefore \frac{1}{2}\dot{x}^2 = -8x^2 + 16$ $\dot{x}^2 = -16x^2 + 32$ $\dot{x}^2 = 16(-x^2 + 2)$ $\therefore \lvert \dot{x} \rvert = 4\sqrt{2 - x^2}$
(c)(ii)	Greatest displacement occurs when $\dot{x} = 0$ $4\sqrt{2 - x^2} = 0$ $\sqrt{2 - x^2} = 0$ $x^2 = 2$ $x = \sqrt{2}$
(c)(iii)	Using $\dot{x}^2 = n^2\left(a^2 - x^2\right)$ and $\dot{x}^2 = 16\left(2 - x^2\right)$ then $n = 4$ and $a = \sqrt{2}$ Using $x = a\cos(nt + \alpha)$ then $x = \sqrt{2}\cos(4t + \alpha)$ When $t = 0, x = 1$ $\therefore 1 = \sqrt{2}\cos(0 + \alpha)$ $\cos\alpha = \frac{1}{\sqrt{2}}$ AND $\dot{x} = -4\sqrt{2}\sin(4t + \alpha)$ When $t = 0, x = 4$ $\therefore 4 = -4\sqrt{2}\sin(0 + \alpha)$ $\sin\alpha = -\frac{1}{\sqrt{2}}$ *If* $\cos\alpha = \frac{1}{\sqrt{2}}$ & $\sin\alpha = -\frac{1}{\sqrt{2}}$ *then* $\alpha = -\frac{\pi}{4}$ $\therefore x = \sqrt{2}\cos(4t - \frac{\pi}{4})$

Question 5

(a)

Step 1: Show true for $n=1$

$LHS = 2\times 1! = 2$

$RHS = 1(1+1) = 2$

True for n=1

Step 2: Assume true for say k

i.e. $2\times 1!+5\times 2!+10\times 3!+\ldots+\left(k^2+1\right)k!=k(k+1)!$

Show true for k+1

i.e. $2\times 1!+5\times 2!+10\times 3!+\ldots+\left(k^2+1\right)k!+\left(\left(k+1\right)^2+1\right)\left(k+1\right)!$

$=\left(k+1\right)\left(k+2\right)!$

$LHS = 2\times 1!+5\times 2!+10\times 3!+\ldots+\left(k^2+1\right)k!+\left(\left(k+1\right)^2+1\right)\left(k+1\right)!$

$= k(k+1)!+\left(\left(k+1\right)^2+1\right)\left(k+1\right)!$

$=\left(k+1\right)!\left(k+\left(k+1\right)^2+1\right)$

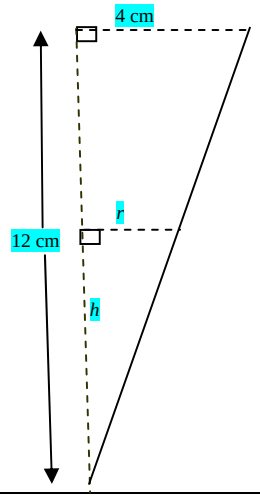
$=\left(k+1\right)!\left(k^2+3k+2\right)$

$=\left(k+1\right)!\left(k+1\right)(k+2)$

$=\left(k+2\right)!\left(k+1\right)$

Step 3: Since true for n=1 (Step 1) then true for n=2 (Step 2)
Since true for n=2 now true for n=3 and so on. Hence by
mathematical induction it is true for all positive integers n.

(b) (i)



From the diagram two similar triangles
can be identified.

$\therefore \frac{r}{h}=\frac{4}{12}$ corresp. sides in the same ratio

$r=\frac{h}{3}$

(b) (ii)

$Use\ \frac{dh}{dt}=\frac{dh}{dV}.\frac{dV}{dt}$

$Now\ V=\frac{1}{3}\pi r^2 h$

$=\frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h$

$=\frac{\pi h^3}{27}$

$\therefore\ \frac{dV}{dh}=\frac{3\pi h^2}{27}=\frac{\pi h^2}{9}$

$\therefore\ \frac{dh}{dV}=\frac{9}{\pi h^2}$

& $\frac{dV}{dt}=3$

$\frac{dh}{dt}=\frac{dh}{dV}.\frac{dV}{dt}$

$=\frac{9}{\pi h^2}.3$

$=\frac{9}{\pi 9^2}.3$ at h=9

$=\frac{1}{3\pi}\ cm/s$

(c)(i)

$f'(x)=2.\frac{1}{\sqrt{1-\left(\sqrt{x^2}\right)}}.\frac{1}{2}x^{-\frac{1}{2}}-\frac{1}{\sqrt{1-(2x-1)^2}}.2$

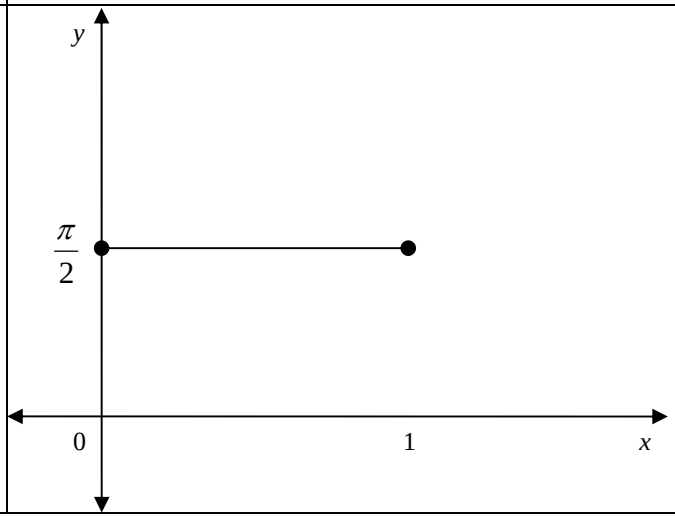
$=\frac{1}{\sqrt{1-x}}.\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{1-\left(4x^2-4x+1\right)}}.2$

$=\frac{1}{\sqrt{1-x}}.\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{4x-4x^2}}.2$

$=\frac{1}{\sqrt{1-x}}.\frac{1}{\sqrt{x}}-.\frac{1}{2}\frac{1}{\sqrt{x}}\frac{1}{\sqrt{1-x}}.2$ NB $x\neq 0$ *or* 1

$=\frac{1}{\sqrt{1-x}}.\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{1-x}}.\frac{1}{\sqrt{x}}$

$=0$ *for* $0<x<1$

(c)(ii)	Since $f'(x) = 0$ for $0 < x < 1$ then $f(x)$ must be a horizontal line. $f(x) = 2\sin^{-1}\sqrt{x} - \sin^{-1}(2x-1)$ $f(0) = 2\sin^{-1}\sqrt{0} - \sin^{-1}(2.0-1)$ $f(0) = 2\times 0 - -\frac{\pi}{2} = \frac{\pi}{2}$ $f(\frac{1}{2}) = 2\sin^{-1}\sqrt{\frac{1}{2}} - \sin^{-1}\left(2.\frac{1}{2}-1\right)$ $f(\frac{1}{2}) = 2\frac{\pi}{4} - \sin^{-1}(0) = \frac{\pi}{2}$ $f(1) = 2\sin^{-1}\sqrt{1} - \sin^{-1}(2.1-1)$ $f(1) = 2\frac{\pi}{2} - \frac{\pi}{2} = \frac{\pi}{2}$ $\therefore f(x) = \frac{\pi}{2}$ *for* $0 \le x \le 1$
(c)(iii)	

Question 6

(a) (i)

Given at $t=0, x=0,\ \dot{x}=V\cos\theta$

and at $t=0, y=5, \dot{y}=V\sin\theta$

$\ddot{x}=0 \qquad \ddot{y}=-10$

$\dot{x}=V\cos\theta \qquad \dot{y}=-10t+V\sin\theta$

$x=Vt\cos\theta \qquad y=-5t^2+Vt\sin\theta+5$

(a) (ii)

If $\tan\theta=\frac{3}{4}$ then $\sin\theta=\frac{3}{5}$ & $\cos\theta=\frac{4}{5}$

From the diagram when x=60 y=0

Using $x=Vt\cos\theta$

the $t=\frac{x}{V\cos\theta}=\frac{60}{V.\frac{4}{5}}=\frac{75}{V}$

Now using $y=-5t^2+Vt\sin\theta+5$

and $t=\frac{75}{V}$ when $y=0$

$$0=-5\left(\frac{75}{V}\right)^2+V.\frac{75}{V}.\frac{3}{5}+5$$

$$=-5\left(\frac{75}{V}\right)^2+\frac{75}{1}.\frac{3}{5}+5$$

$$=-5\left(\frac{75}{V}\right)^2+50$$

$$-50=-5\frac{75^2}{V^2}$$

$$V^2=\frac{75^2}{10}$$

$$V=\sqrt{\frac{75^2}{10}}=\frac{75}{\sqrt{10}}\approx 23.7$$

(a) (iii)

Achieves maximum height when $\dot{y} = 0$

Now $\dot{y} = -10t + V\sin\theta$

$$\therefore \quad 0 = -10t + \frac{75}{\sqrt{10}} \cdot \frac{3}{5}$$

$$10t = \frac{45}{\sqrt{10}}$$

$$t = \frac{45}{10\sqrt{10}} = \frac{9}{2\sqrt{10}}$$

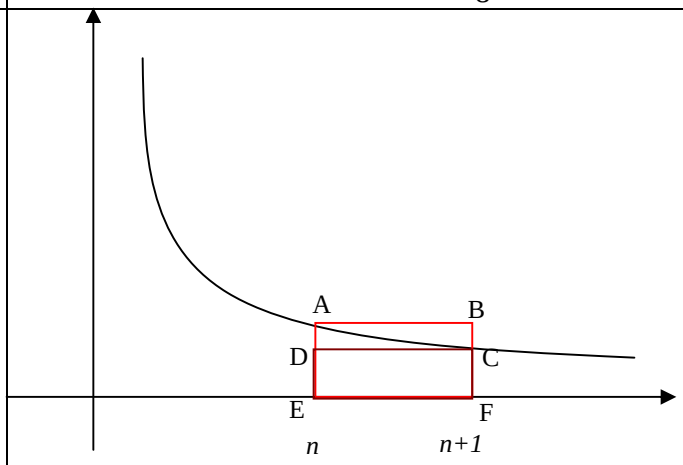
Sub this value for t into y

$$\therefore y = -5\left(\frac{9}{2\sqrt{10}}\right)^2 + \frac{75}{\sqrt{10}} \cdot \frac{9}{2\sqrt{10}} \cdot \frac{3}{5} + 5$$

$$= \frac{121}{8}$$

$\therefore$ the maximum height reached is $\frac{121}{8} m$

(b) (i)



$\int_n^{n+1} \frac{1}{x} dx$ represents the area under the curve between n and n+1

In the diagram this would be ACFE

$\frac{1}{\text{n+1}}$ represents the area of the small rectangle DCFE

as CF = $f(n+1) = \frac{1}{\text{n+1}}$

$\frac{1}{\text{n}}$ represents the area of the bigger rectangle ABFE

as AE = $f(n) = \frac{1}{\text{n}}$

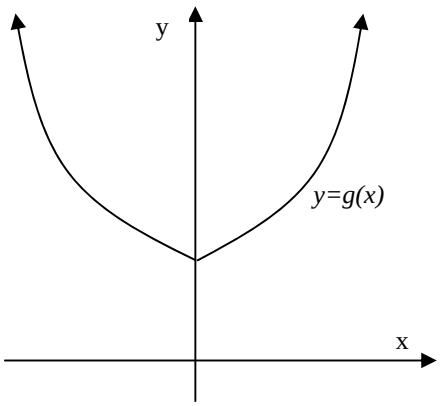
DCFE < ACFE < ABFE

$$\therefore \frac{1}{\text{n+1}} < \int_n^{n+1} \frac{1}{x} dx < \frac{1}{\text{n}}$$

(b) (ii)	$\frac{1}{n+1} < \int_n^{n+1} \frac{1}{x}\,dx < \frac{1}{n}$ $\therefore \frac{1}{n+1} < \left[\ln x\right]_n^{n+1} < \frac{1}{n}$ $\therefore \frac{1}{n+1} < \ln(n+1) - \ln n < \frac{1}{n}$ $\therefore \frac{1}{n+1} < \ln\frac{(n+1)}{n} < \frac{1}{n}$ $\therefore \frac{1}{n+1} < \ln(1+\frac{1}{n}) < \frac{1}{n}$ $\therefore e^{\frac{1}{n+1}} < e^{\ln(1+\frac{1}{n})} < e^{\frac{1}{n}}$ $\therefore e^{\frac{1}{n+1}} < 1+\frac{1}{n} < e^{\frac{1}{n}}$ $\therefore e^{\frac{1}{n+1}} < 1+\frac{1}{n}$ and $1+\frac{1}{n} < e^{\frac{1}{n}}$ $\therefore e < \left(1+\frac{1}{n}\right)^{n+1}$ and $\left(1+\frac{1}{n}\right)^{n} < e$ $\therefore \left(1+\frac{1}{n}\right)^{n} < e < \left(1+\frac{1}{n}\right)^{n+1}$

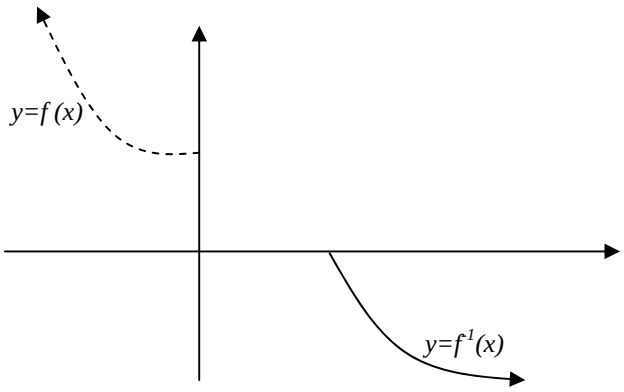
Question 7

(a) (i)



It does not have an inverse as it does not pass the horizontal line test. Over it's domain it is both monotonic increasing and monotonic decreasing

(a) (ii)



(a) (iii)

$x = e^y + \frac{1}{e^y}$ for y ≤ 0 and x ≥ 2

$xe^y = e^{2y} + 1$

$0 = e^{2y} - xe^y + 1$

Let $u = e^y$

$\therefore 0 = u^2 - xu + 1$

$\therefore u = \frac{--x \pm \sqrt{(-x)^2 - 4 \times 1 \times 1}}{2 \times 1}$

$u = \frac{x \pm \sqrt{x^2 - 4}}{2}$

i.e. $e^y = \frac{x \pm \sqrt{x^2 - 4}}{2}$

$\therefore y = \ln\left(\frac{x \pm \sqrt{x^2 - 4}}{2}\right)$

However $y = \ln\left(\frac{x - \sqrt{x^2 - 4}}{2}\right)$

because for $x \geq 2$ $y \leq 0$ which are the conditions for the domain and range

(b) (i)	$(1+x)^n = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots + c_nx^n \ldots\ldots\ldots\ldots\ldots\ldots(1)$ $\frac{d}{dx}(1+x)^n = \frac{d}{dx}\left(c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots + c_nx^n\right)$ $n(1+x)^{n-1} = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \ldots + nc_nx^{n-1} \ldots\ldots(2)$ Let $x = 1$ $(2)^n = c_0 + c_1 + c_2 + c_3 + \ldots \quad + c_n \quad from\,(1)$ $n(2)^{n-1} = \quad c_1 + 2c_2 + 3c_3 + 4c_4 + \ldots + nc_n \quad from\,(2)$ $\therefore (2)^n + n(2)^{n-1} = c_0 + 2c_1 + 3c_2 + 4c_3 + \ldots \quad + (n+1)c_n \quad added$ $i.e.(2)^{n-1}(n+2) = c_0 + 2c_1 + 3c_2 + 4c_3 + \ldots \quad + (n+1)c_n$

(b) (ii)

$$(1+x)^n = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots + c_nx^n$$

$$\int (1+x)^n\,dx = \int \left(c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots + c_nx^n\right)dx$$

$$\frac{(1+x)^{n+1}}{n+1} = c_0x + \frac{c_1x^2}{2} + \frac{c_2x^3}{3} + \frac{c_3x^4}{4} + \ldots + \frac{c_nx^{n+1}}{n+1} + C$$

Let $x = 0$

$$\frac{1}{n+1} = 0 + 0 + 0 + 0 + \ldots. + 0 + C$$

$$\therefore C = \frac{1}{n+1}$$

$$\therefore \frac{(1+x)^{n+1}}{n+1} = c_0x + \frac{c_1x^2}{2} + \frac{c_2x^3}{3} + \frac{c_3x^4}{4} + \ldots + \frac{c_nx^{n+1}}{n+1} + \frac{1}{n+1}$$

$$\int \frac{(1+x)^{n+1}}{n+1}dx = \int (c_0x + \frac{c_1x^2}{2} + \frac{c_2x^3}{3} + \frac{c_3x^4}{4} + \ldots + \frac{c_nx^{n+1}}{n+1} + \frac{1}{n+1})$$

$$\frac{(1+x)^{n+2}}{(n+1)(n+2)} = \frac{c_0x^2}{2} + \frac{c_1x^3}{2.3} + \frac{c_2x^4}{3.4} + \frac{c_3x^5}{4.5} + \ldots + \frac{c_nx^{n+2}}{(n+1)(n+2)} + \frac{1}{n+1}x + C_1$$

Let $x = 0$

$$\frac{1}{(n+1)(n+2)} = 0 + 0 + 0 + 0 + \ldots. + 0 + C_1$$

$$\therefore C_1 = \frac{1}{(n+1)(n+2)}$$

$$\therefore \frac{(1+x)^{n+2}}{(n+1)(n+2)} = \frac{c_0x^2}{2} + \frac{c_1x^3}{2.3} + \frac{c_2x^4}{3.4} + \ldots + \frac{c_nx^{n+2}}{(n+1)(n+2)} + \frac{1}{n+1}x + \frac{1}{(n+1)(n+2)}$$

Let $x = -1$

$$\therefore 0 = \frac{c_0}{2} - \frac{c_1}{2.3} + \frac{c_2}{3.4} - \ldots + (-1)^{n+2}\frac{c_n}{(n+1)(n+2)} - \frac{1}{n+1} + \frac{1}{(n+1)(n+2)}$$

$$\therefore \frac{c_0}{2} - \frac{c_1}{2.3} + \frac{c_2}{3.4} - \ldots + (-1)^{n+2}\frac{c_n}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{(n+1)(n+2)}$$

$$= \frac{(n+2)-1}{(n+1)(n+2)}$$

$$= \frac{(n+1)}{(n+1)(n+2)}$$

$$= \frac{1}{(n+2)}$$