2000 HIGHER SCHOOL CERTIFICATE SOLUTIONS

3 UNIT (ADDITIONAL) AND 3/4 UNIT (COMMON) MATHEMATICS

QUESTION 1

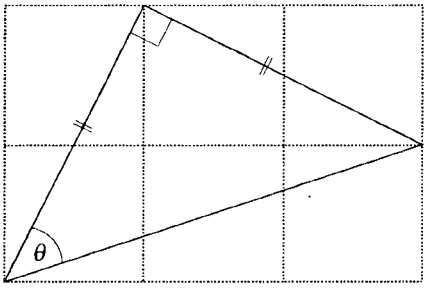
(a) $\dfrac{d}{dx}x\sin^{-1}x = \sin^{-1}x + \dfrac{x}{\sqrt{1-x^2}}$.

(b) $m_1 = 2, m_2 = \frac{1}{3}$.

$$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right|$$
$$= \frac{2-\frac{1}{3}}{1+2\times\frac{1}{3}}$$
$$= 1.$$

$\therefore$ The acute angle is $45°$ $\left(\frac{\pi}{4}\text{ radians}\right)$.

Alternative solution:



The sketch shows two lines with gradients 2 and $\frac{1}{3}$. By considering the triangle formed (right-angled isosceles), the angle between the two lines, θ, is $45°$.

(c) **Method 1:**

By the factor theorem, P(3) = 0 when $x - 3$ is a factor.

$$\text{P}(3) = 27 - 9k + 6 = 0$$
$$9k = 33$$
$$k = 3\frac{2}{3}.$$

Method 2:

$$\text{P}(x) = (x-3)\left(x^2+3x-2\right)$$
$$= x^3 - 11x + 6.$$

Equating coefficients, $k = \frac{11}{3} = 3\frac{2}{3}$.

Method 3:

$\text{P}(x) = x^3 - 3kx + 6$

Let roots be $3, \alpha, \beta$.

$$\alpha+\beta+3 = 0 \quad ①$$
$$3\alpha+3\beta+\alpha\beta = -3k \quad ②$$
$$3\alpha\beta = -6 \quad ③$$

From ①: $\alpha+\beta = -3$

From ③: $\alpha\beta = -2$

In ②: $3(\alpha+\beta)+\alpha\beta = -3k$

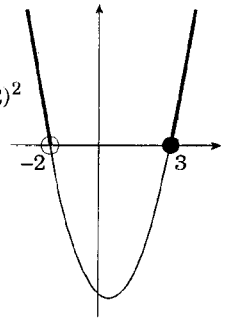
$$3(-3)-2 = -3k$$
$$k = \frac{11}{3}.$$

(d) $\displaystyle\int_0^{\sqrt{3}} \frac{4}{x^2+9}\,dx = \left[\frac{4}{3}\tan^{-1}\frac{x}{3}\right]_0^{\sqrt{3}}$ (standard integral)

$$= \frac{4}{3}\left(\tan^{-1}\frac{\sqrt{3}}{3} - \tan^{-1}0\right)$$
$$= \frac{4}{3}\left(\frac{\pi}{6} - 0\right)$$
$$= \frac{2\pi}{9}.$$

(e) **Method 1:**

$$\frac{5}{x+2} \le 1, \quad x \ne -2$$
$$\therefore \quad 5(x+2) \le (x+2)^2$$
$$(x+2)^2 - 5(x+2) \ge 0$$
$$\therefore \quad (x+2)(x+2-5) \ge 0$$
$$(x+2)(x-3) \ge 0.$$
$$\therefore \quad x < -2 \text{ or } x \ge 3.$$



Method 2:

$$\frac{5}{x+2} \le 1, \quad x \ne -2$$

Consider $\dfrac{5}{x+2} = 1$

$$\therefore \quad 5 = x+2$$
$$x = 3.$$



$\therefore$ $x < -2$ or $x \ge 3$.

Method 3:

When $x + 2 > 0$, that is, $x > -2$,

$$5 \le x+2,$$

that is, $x \ge 3$.

When $x + 2 < 0$, that is, $x < -2$,

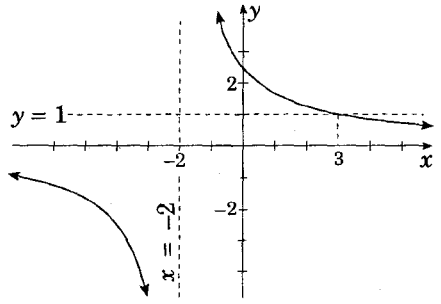
$$5 \ge x+2,$$

that is, $x \le 3$. Hence $x < -2$.

$\therefore$ Solution is $x < -2$ or $x \ge 3$.

Method 4:

Graph $y = \dfrac{5}{x+2}$ and $y = 1$.



Point of intersection when $x = 3$
(see Method 2).
From the graph, $x < -2$ or $x \ge 3$.

QUESTION 2

(a) 'O' is repeated 3 times.
The other letters are unique.

$\therefore$ The number of arrangements $= \dfrac{9!}{3!}$
$= 60\,480.$

(b) $\left(5+2x^2\right)^7 = \sum_{r=0}^{7} {}^7C_r 5^{7-r}\left(2x^2\right)^r.$

For a term in x^6 we require $2r = 6$, $\therefore r = 3$.

$\therefore$ The coefficient of x^6 is ${}^7C_3\, 5^{7-3}\, 2^3 = 175\,000.$

(c) $\cos 2\theta = \sin\theta,$

$$\therefore \quad 1-2\sin^2\theta = \sin\theta$$
$$2\sin^2\theta + \sin\theta - 1 = 0$$
$$(2\sin\theta - 1)(\sin\theta + 1) = 0$$
$$\sin\theta = \frac{1}{2} \text{ or } -1.$$

$\therefore \theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{3\pi}{2}.$

(d) $u = 2+x$, $x = u-2$, $\therefore \dfrac{dx}{du} = 1.$

$$\int \frac{x}{\sqrt{2+x}}\,dx = \int \frac{u-2}{u^{\frac{1}{2}}}\,du$$
$$= \int \left(u^{\frac{1}{2}} - 2u^{-\frac{1}{2}}\right)du$$
$$= \frac{2u^{\frac{3}{2}}}{3} - 4u^{\frac{1}{2}} + C$$
$$= \frac{2}{3}(2+x)^{\frac{3}{2}} - 4\sqrt{2+x} + C$$
$$= \frac{2}{3}\sqrt{2+x}\left(2+x-\frac{3}{2}\times 4\right) + C$$
$$= \frac{2}{3}\sqrt{2+x}\,(x-4) + C.$$

QUESTION 3

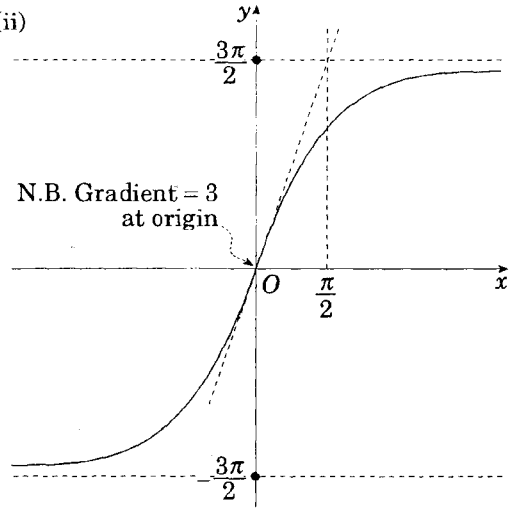
(a) $f'(a) = \lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$

$$= \lim_{h\to 0} \frac{(a+h)^3 - a^3}{h}$$
$$= \lim_{h\to 0} \frac{a^3 + 3a^2h + 3ah^2 + h^3 - a^3}{h}$$
$$= \lim_{h\to 0} \frac{h\left(3a^2 + 3ah + h^2\right)}{h}$$
$$= \lim_{h\to 0} \left(3a^2 + 3ah + h^2\right)$$
$$= 3a^2.$$

(b) $f(x) = 3\tan^{-1}x.$

(i) Range is $-\dfrac{3\pi}{2} \le y \le \dfrac{3\pi}{2}.$

(ii)



(iii) $f'(x) = \dfrac{3}{1+x^2}$

$$f'\left(\frac{1}{\sqrt{3}}\right) = \frac{3}{1+\frac{1}{3}} = 2\frac{1}{4}.$$

Therefore the gradient of the tangent at $x = \dfrac{1}{\sqrt{3}}$ is $2\frac{1}{4}$.

(c) (i) From ΔOTB, $\dfrac{h}{OB} = \tan 30° = \dfrac{1}{\sqrt{3}},$

$\therefore OB = \sqrt{3}h.$

(Other answers are possible, such as $OB = \sqrt{100^2 + h^2}$.)

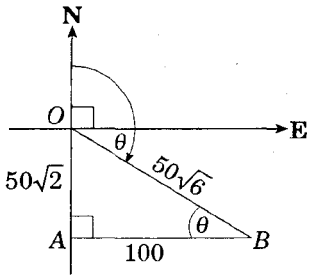
(ii) From ΔOAT, $\dfrac{OA}{h} = \tan 45° = 1,$

$\therefore OA = h.$

From ΔOAB, $OB^2 = AO^2 + AB^2$

$$3h^2 = h^2 + 100^2$$
$$2h^2 = 100^2$$
$$\therefore \quad h = 50\sqrt{2}.$$

(iii)



$$\sin\theta = \frac{OA}{OB}$$
$$= \frac{50\sqrt{2}}{50\sqrt{6}}$$
$$= \frac{1}{\sqrt{3}}.$$

$\therefore \quad \theta = 35^\circ 16'.$

Therefore the bearing is $125^\circ 16'$.

QUESTION 4

(a) Let $S(n)$ be the statement

$$1+3+6+\cdots+\frac{1}{2}n(n+1) = \frac{1}{6}n(n+1)(n+2).$$

When $n = 1$, LHS $= 1$

RHS $= \frac{1}{6}\times 1\times 2\times 3 = 1.$

$\therefore$ LHS = RHS.

$\therefore$ $S(1)$ is true.

Assume $S(k)$ is true. That is,

$$1+3+6+\cdots+\frac{1}{2}k(k+1) = \frac{1}{6}k(k+1)(k+2).$$

Now

$$1+3+6+\cdots+\frac{1}{2}k(k+1)+\frac{1}{2}(k+1)(k+1+1)$$
$$= \frac{1}{6}k(k+1)(k+2)+\frac{1}{2}(k+1)(k+1+1)$$
(by assumption)
$$= \frac{(k+1)(k+2)}{6}(k+3)$$
$$= \frac{1}{6}(k+1)(k+1+1)(k+1+2).$$

This is $S(k+1)$.

$\therefore$ If $S(k)$ is true, then $S(k+1)$ is true.

But $S(1)$ is true, hence $S(2)$ is true, hence $S(3)$ is true and so on.

By the principle of mathematical induction, the result is true for all positive integral values of n.

(b) Let $f(r) = (1+r)\left[(1+r)^{24}-1\right]-50r$

$$= (1+r)^{25}-(1+r)-50r$$
$$= (1+r)^{25}-1-51r.$$

$$f'(r) = 25(1+r)^{24}-51$$

$r_1 = 0{\cdot}06,$

then $r_2 = r_1 - \dfrac{f(r_1)}{f'(r_1)}$

$$= 0{\cdot}06 - \frac{(1{\cdot}06)^{25}-1-51\times 0{\cdot}06}{25(1{\cdot}06)^{24}-51}$$
$$= 0{\cdot}055\,38\ldots$$
$$= 0{\cdot}055.$$

(c) $P(x) = x^3+px^2+qx+r.$

(i) Sum of roots $= -\dfrac{b}{a}.$

$\therefore \quad \sqrt{k}+\left(-\sqrt{k}\right)+\alpha = -p$

$\therefore \quad \alpha+p = 0.$

(ii) Product of roots $= -\dfrac{d}{a}.$

$\therefore \quad \sqrt{k}\times\left(-\sqrt{k}\right)\times\alpha = -r$

$\therefore \quad -k\alpha = -r$

$k\alpha = r.$

(iii) Product of roots in pairs $= \dfrac{c}{a}.$

$\therefore \quad \sqrt{k}\times\left(-\sqrt{k}\right)+\sqrt{k}\times\alpha+\left(-\sqrt{k}\right)\times\alpha = q$

$\therefore \quad -k = q.$

Since $\alpha = -p$, from (i), then $k\alpha = r$ from (ii) becomes $(-q)\times(-p) = r$, that is, $pq = r.$

(d) Period $T = \dfrac{2\pi}{n} = 4\pi,$

$\therefore \quad n = \frac{1}{2}.$

Amplitude $a = 3.$

Since $v^2 = n^2\left(a^2-x^2\right)$ (putting origin at O)

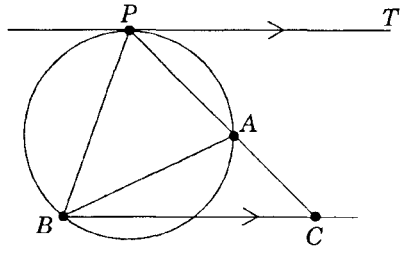
$$v^2 = \frac{1}{4}\left(9-x^2\right).$$

When $x = 0$, at O, $v^2 = \dfrac{9}{4}.$

$\therefore$ Speed $= \dfrac{3}{2}$ cm s^{-1}.

QUESTION 5

(a)



(i) $\angle PBA = \angle TPA$ ($\angle$ between tangent and chord equals $\angle$ in alternate segment).

$\angle TPA = \angle PCB$ (alternate $\angle$s, $PT \| BC$),

$\therefore \quad \angle PBA = \angle PCB.$

(ii) In ΔPBA and ΔPCB,

$\angle APB = \angle BPC$ (common)

$\angle PBA = \angle PCB$, from (i).

Therefore the triangles are equiangular, and hence similar.

$\therefore \quad \frac{PB}{PC} = \frac{PA}{PB}$ (corresponding sides are in the same ratio)

$\therefore \quad PB^2 = PA \times PC.$

(b) $f(x) = \frac{x}{x+2}$, defined for all $x \neq -2$.

(i) $f'(x) = \frac{(x+2)\times 1 - x \times 1}{(x+2)^2}$

$= \frac{2}{(x+2)^2} > 0$ for all $x \neq -2$,

since $(x+2)^2 > 0$ for all $x \neq -2$.

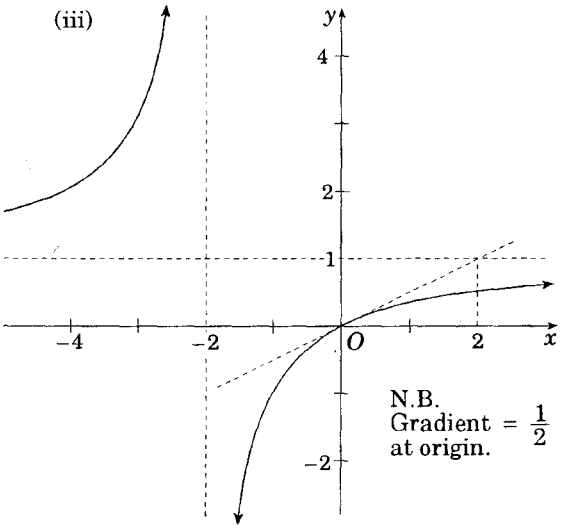
(ii) $f(x) = \frac{x+2-2}{x+2}$

$= 1 - \frac{2}{x+2}.$

As $x \to \pm\infty$, $\frac{2}{x+2} \to 0$.

Therefore the horizontal asymptote is $y = 1$.

(iii)



(iv) $f(x)$ is a one-to-one increasing function (it satisfies the horizontal-line test).

(v) $y = \frac{x}{x+2}.$

$\therefore$ The inverse is $x = \frac{y}{y+2}.$

$\therefore \quad xy + 2x = y$

$2x = y(1-x)$

$y = \frac{2x}{1-x}$

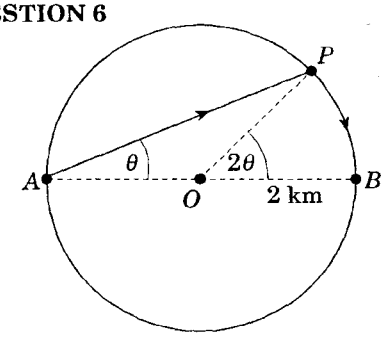
$\therefore \quad f^{-1}(x) = \frac{2x}{1-x}.$

(vi) Domain of $f^{-1}(x)$ is the range of $f(x)$.

That is, all real x, $x \neq 1$.

QUESTION 6

(a)



(i) $\angle APB = \frac{\pi}{2}$ ($\angle$ in a semicircle).

$\therefore \frac{AP}{AB} = \cos\theta$

$\therefore \quad AP = 4\cos\theta.$

Arc $PB = 2.2\theta = 4\theta.$

Time from A to B

= time for AP + time for PB.

That is, $T = \frac{AP}{3} + \frac{PB}{4}$

$= \frac{4\cos\theta}{3} + \theta$

$= \frac{1}{3}(4\cos\theta + 3\theta).$

(ii) $\frac{dT}{d\theta} = \frac{1}{3}(-4\sin\theta + 3)$

$= 0$ when $\sin\theta = \frac{3}{4}.$

$\therefore \quad \theta = \sin^{-1}\left(\frac{3}{4}\right)$

$= 0{\cdot}848$ radians $(= 48° 35')$.

(iii) $\frac{d^2T}{d\theta^2} = \frac{1}{3}(-4\cos\theta) < 0$ for θ acute.

Therefore this is a maximum stationary point, and so not the minimum.

Test the end points.

That is, row direct to B.

Time $= \frac{4}{3} = 1\frac{1}{3}$ hours.

Walk round the lake from A to B.

Time $= \frac{2\pi}{4} = 1{\cdot}57$ hours.

$\therefore$ Pat should row directly across the lake to B to minimise the time.

(b) (i) No. of ways for 2 spades and 4 clubs

$= {}^{13}C_2 \times {}^{13}C_4 = 55\,770.$

(ii) No. of ways with 5 of the same suit

$= 4 \times {}^{13}C_5 \times {}^{39}C_1 = 200\,772.$

No. of ways with 6 of the same suit

$= 4 \times {}^{13}C_6 = 6864.$

$\therefore$ Total number with at least 5 of the same suit $= 200\,772 + 6864 = 207\,636.$

QUESTION 7

(a) (i) $F = Au^3 + \frac{B}{u}$.

The maximum period of flight implies the minimum fuel used per hour.

$$\frac{dF}{du} = 3Au^2 - \frac{B}{u^2}.$$

$$\frac{d^2F}{du^2} = 6Au + \frac{2B}{u^3} > 0,$$

since $u > 0$ and A and B are positive.

Therefore we have a minimum when

$$\frac{dF}{du} = 0,$$

that is, when $3Au^2 = \frac{B}{u^2}$

$$u^4 = \frac{B}{3A}$$

$$u = \left(\frac{B}{3A}\right)^{\frac{1}{4}}.$$

(ii) Let the distance be s and the time t.

$$s = ut$$

$$= u\frac{k}{F} \quad \text{(where } k \text{ is a positive constant, the amount of fuel)}$$

$$= \frac{uk}{Au^3 + \frac{B}{u}}$$

$$= \frac{u^2k}{Au^4 + B}.$$

$$\frac{ds}{du} = \frac{2uk(Au^4 + B) - u^2k \cdot 4Au^3}{(Au^4 + B)^2}$$

$$= \frac{2Aku^5 + 2Bku - 4Aku^5}{(Au^4 + B)^2}$$

$$= \frac{2Bku - 2Aku^5}{(Au^4 + B)^2}$$

$$= \frac{2ku(B - Au^4)}{(Au^4 + B)^2}.$$

For a maximum s, $\frac{ds}{du} = 0$.

$\therefore\ u = 0$ or $u^4 = \frac{B}{A}$.

When $u = 0$, s is obviously a minimum value,

$\therefore$ maximum occurs when $u = \left(\frac{B}{A}\right)^{\frac{1}{4}}$.

$$\frac{\text{New speed}}{\text{Old speed}} = \frac{\left(\frac{B}{A}\right)^{\frac{1}{4}}}{\left(\frac{B}{3A}\right)^{\frac{1}{4}}}$$

$$= 3^{\frac{1}{4}}$$

$$= 1{\cdot}3162$$

$$= 132\% \text{ (nearest per cent)}.$$

Therefore the speed for maximum distance is approximately 32% faster than the speed for maximum time.

(b) (i) Given $x = Vt\cos\theta$

$$\therefore\ t = \frac{x}{V\cos\theta}$$

$$y = Vt\sin\theta - \frac{1}{2}gt^2$$

$$= \frac{V \cdot x\sin\theta}{V\cos\theta} - \frac{1}{2}g\frac{x^2}{V^2\cos^2\theta}$$

$$= x\tan\theta - x^2\sec^2\theta\left(\frac{g}{2V^2}\right)$$

$$= x\tan\theta - x^2\sec^2\theta, \text{ since } \frac{2V^2}{g} = 1.$$

(ii) On the inclined plane,

$$x = r\cos\alpha,\ y = r\sin\alpha.$$

$$\therefore\ r\sin\alpha = r\cos\alpha\tan\theta - \frac{r^2\cos^2\alpha}{\cos^2\theta}.$$

$$\sin\alpha\cos^2\theta$$

$$= \cos\alpha\sin\theta\cos\theta - r\cos^2\alpha \quad (\text{since } r \neq 0)$$

$$\therefore\ r\cos^2\alpha = \cos\theta(\sin\alpha\cos\theta - \cos\alpha\sin\theta)$$

$$\therefore\ r = \frac{\cos\theta \cdot \sin(\theta - \alpha)}{\cos^2\alpha}.$$

(iii) Using the given identity with $A = \theta - \alpha$ and $B = \theta$, we have

$$r = \frac{\sin(\theta - \alpha)\cos\theta}{\cos^2\alpha}$$

$$= \frac{\sin(2\theta - \alpha) + \sin(-\alpha)}{2\cos^2\alpha}.$$

This will be a maximum when $\sin(2\theta - \alpha) = 1$.

$$\left(\text{That is, when } 2\theta - \alpha = \frac{\pi}{2}.\right)$$

So the maximum range is

$$R = \frac{1 + \sin(-\alpha)}{2\cos^2\alpha}$$

$$= \frac{1 - \sin\alpha}{2(1 - \sin^2\alpha)}$$

$$= \frac{1 - \sin\alpha}{2(1 - \sin\alpha)(1 + \sin\alpha)}$$

$$= \frac{1}{2(1 + \sin\alpha)}.$$

(iv) **Method 1**:

Let m_O and m_T be the slopes of the tangents at O and T respectively.

$$y = x\tan\theta - x^2\sec^2\theta$$

$$y' = \tan\theta - 2x\sec^2\theta.$$

At O, $x = 0$, so $m_O = \tan\theta$.

Equation of OT is $y = x\tan\alpha$,

so T satisfies $x\tan\alpha = x\tan\theta - x^2\sec^2\theta$.

$$\therefore\ x\sec^2\theta = \tan\theta - \tan\alpha \quad (x \neq 0 \text{ at } T)$$

$$x = \frac{\tan\theta - \tan\alpha}{\sec^2\theta}.$$

$$\therefore \quad m_T = \tan\theta - 2\left(\frac{\tan\theta - \tan\alpha}{\sec^2\theta}\right).\sec^2\theta$$
$$= 2\tan\alpha - \tan\theta$$
$$= 2\tan\left(2\theta - \frac{\pi}{2}\right) - \tan\theta \quad \text{(from (iii))}$$
$$= -2\cot 2\theta - \tan\theta$$
$$= -2\left(\frac{1-\tan^2\theta}{2\tan\theta}\right) - \tan\theta$$
$$= -\frac{1}{\tan\theta} + \tan\theta - \tan\theta$$
$$= -\frac{1}{\tan\theta}.$$
$$m_O m_T = \tan\theta \times \left(-\frac{1}{\tan\theta}\right)$$
$$= -1.$$

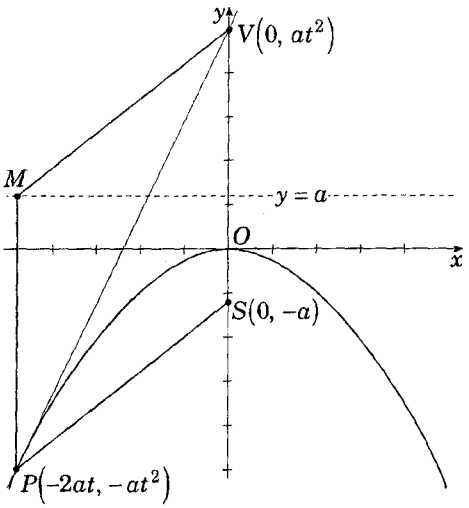
Therefore the tangents are perpendicular.

Method 2:
The endpoints of a focal chord have tangents which are perpendicular, so it is enough to show that for maximum range OT is a focal chord.

Method 2a (focal chord from geometry):
The following diagram shows a general parabola $x^2 = -4ay$ with focus $S(0, -a)$. The tangent at $P\left(y = tx + at^2\right)$ intersects the y axis at $V\left(0, at^2\right)$.

Therefore $PM = SV\ \left(= at^2 + a\right)$.

Also, $PM = PS$ by definition of a parabola.
Therefore $PMVS$ is a rhombus, and PV bisects $\angle MPS$.



This only works because PS is a focal chord. For any other point S on the y axis, $\angle VPS$ would be different.
So if a tangent bisects the angle between the vertical and a chord, that chord must be a focal chord.

In part (iii) we showed that the maximum range occurs when
$$2\theta - \alpha = \frac{\pi}{2}.$$
$$\therefore \quad \theta - \alpha = \frac{\pi}{2} - \theta.$$

This means that the tangent at O (with angle θ) bisects the vertical (with angle $\frac{\pi}{2}$) and OT (with angle α). Hence OT is a focal chord as required.

Method 2b (focal chord using algebra):
$$y = x\tan\theta - x^2\sec^2\theta$$
$$\therefore \quad \cos^2\theta . y = x\sin\theta\cos\theta - x^2.$$
$$\left(x - \frac{\sin\theta\cos\theta}{2}\right)^2 = \frac{\sin^2\theta\cos^2\theta}{4} - \cos^2\theta . y$$
$$= -\cos^2\theta\left(y - \frac{\sin^2\theta}{4}\right).$$

Focal length $= \frac{\cos^2\theta}{4}$.

Vertex $= \left(\frac{\sin\theta\cos\theta}{2}, \frac{\sin^2\theta}{4}\right)$.

Focus $F = \left(\frac{\sin\theta\cos\theta}{2}, \frac{\sin^2\theta - \cos^2\theta}{4}\right)$.

Slope of $OF = \frac{\sin^2\theta - \cos^2\theta}{2\sin\theta\cos\theta}$
$$= \frac{1}{2}\left(\tan\theta - \frac{1}{\tan\theta}\right).$$

Now for maximum range, $2\theta - \alpha = \frac{\pi}{2}$.
$$\tan 2\theta = \tan\left(\alpha + \frac{\pi}{2}\right)$$
$$= -\frac{1}{\tan\alpha}$$
$$\therefore \quad \tan\alpha = -\frac{1}{\tan 2\theta}$$
$$= -\frac{1-\tan^2\theta}{2\tan\theta}$$
$$= \frac{1}{2}\left(\tan\theta - \frac{1}{\tan\theta}\right).$$

Therefore the slope of OF equals the slope of OT, and so OT is a focal chord, as required.

END OF 3 UNIT (ADDITIONAL) AND 3/4 UNIT (COMMON) MATHEMATICS SOLUTIONS