2000 HSC Full Solutions

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QUESTION 1

(a)

$$\frac{d}{dx}(x\sin^{-1}x)=\sin^{-1}x+\frac{x}{\sqrt{1-x^2}}$$

(b)

Let α be the angle between the two lines

$m_1=2$ & $m_2=\frac{1}{3}$

Now

$$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

$$= \left|\frac{2-\frac{1}{3}}{1+2\times\frac{1}{3}}\right|$$

$$=1$$

$$\therefore \theta = 45^0$$

(c)

If $(x-3)$ is a factor then $P(3)=0$

$P(3)=3^3-3k.3+6$

$0=33-9k$

$k=\frac{11}{3}$

(d)

$$\int_0^{\sqrt{3}}\frac{4}{x^2+9}dx=\frac{4}{3}\int_0^{\sqrt{3}}\frac{3}{x^2+9}dx$$

$$=\frac{4}{3}\left[\tan^{-1}\frac{x}{3}\right]_0^{\sqrt{3}}$$

$$=\frac{4}{3}\tan^{-1}\frac{\sqrt{3}}{3}=\frac{4}{3}\times\frac{\pi}{6}=\frac{2\pi}{9}$$

(e)

$$\frac{5}{x-2}\le 1$$

$$(x-2)^2\frac{5}{x-2}\le 1.(x-2)^2$$

$$5(x-2)\le(x-2)^2$$

$$5x+10\le x^2-4x+4$$

$$0\le x^2-x-6$$

$$0\le(x-3)(x+2)$$

$$x<-2 \ \ or \ \ x\ge 3$$

note $x\neq 2$

Question 2

(a)

9 letters 3 alike

$$\frac{9!}{3!} = 9\times 8\times 7\times 6\times 5\times 4$$

$$= 60480$$

(b)

General term: ${}^{n}C_{r}(a)^{n-r}(b)^{r}$

${}^{7}C_{r}(5)^{7-r}(2x^{2})^{r}$

the term with x^{6} will be with $2r = 3$ *i.e.* $r = 3$

${}^{7}C_{3}(5)^{7-3}(2x^{2})^{3}$

so the coefficient will be

${}^{7}C_{3}.5^{4}2^{3} = 175000$

(c)

$\cos 2\theta = \sin\theta, \quad 0 \le \theta \le 2\pi$

$1-2\sin^{2}\theta = \sin\theta$

$0 = 2\sin^{2}\theta + \sin\theta - 1$

$0 = (2\sin\theta - 1)(\sin\theta + 1)$

$\therefore (2\sin\theta - 1) = 0 \quad or \quad (\sin\theta + 1) = 0$

$\sin\theta = \frac{1}{2} \qquad or \quad \sin\theta = -1$

$\therefore \theta = \frac{\pi}{6}, \frac{5\pi}{6} \; or \; \frac{3\pi}{2} \qquad 0 \le \theta \le 2\pi$

(d)

Let $u = 2 + x$ *and* $x = u - 2$

$\frac{du}{dx} = 1$

$du = dx$

$Now \int \frac{x}{\sqrt{2+x}}\,dx$

$= \int \frac{u-2}{\sqrt{2+u-2}}\,du$

$= \int \frac{u-2}{\sqrt{u}}\,du$

$= \int u^{\frac{1}{2}} - 2u^{-\frac{1}{2}}\,du$

$= \frac{2}{3}u^{\frac{3}{2}} - 4u^{\frac{1}{2}} + C$

$= 2u^{\frac{1}{2}}(\frac{1}{3}u - 2) + C$

$= 2\sqrt{2+x}\left(\frac{2+x}{3} - 2\right) + C$

$= 2\sqrt{2+x}\left(\frac{2+x}{3} - \frac{6}{3}\right) + C$

$= \frac{2}{3}\sqrt{2+x}\,(x-4) + C$

Question 3

(a) (i)

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

$$f'(a)=\lim_{h\to 0}\frac{(a+h)^3-a^3}{h}$$

$$=\lim_{h\to 0}\frac{a^3+3a^2h+3ah^2+h^3-a^3}{h}$$

$$=\lim_{h\to 0}\frac{3a^2h+3ah^2+h^3}{h}$$

$$=\lim_{h\to 0}\frac{h(3a^2+3ah+h^2)}{h}$$

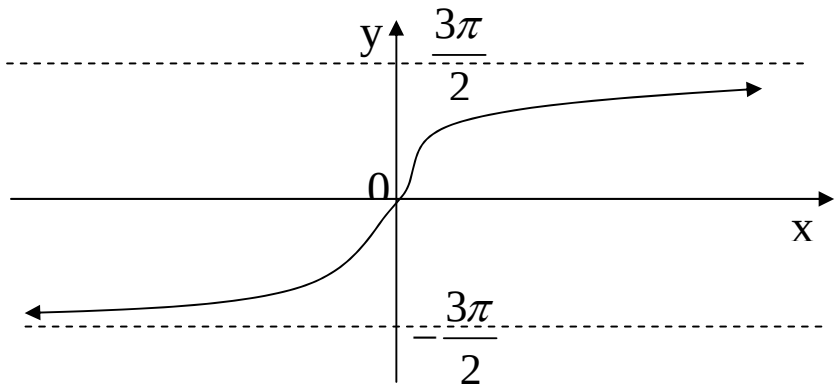
$$=\lim_{h\to 0}3a^2+3ah+h^2$$

$$=3a^2$$

(b) (i)

$$Range: \frac{3\pi}{2}\le y\le\frac{3\pi}{2}$$

(b) (ii)



(b) (iii)

$$y=3\tan^{-1}x$$

$$\frac{dy}{dx}=\frac{3}{1+x^2}$$

$$At\ x=\frac{1}{\sqrt{3}}$$

$$\frac{dy}{dx}=\frac{3}{1+\left(\frac{1}{\sqrt{3}}\right)^2}$$

$$\frac{dy}{dx}=\frac{3}{\frac{4}{3}}=\frac{9}{4}$$

(c) (i)

$$\tan 30^\circ=\frac{h}{OB},\therefore OB=h\cot 30^\circ=\sqrt{3}h$$

(c) (ii)	(ii) $\tan 45° = \frac{h}{OA}, \therefore OA = h \tan 45° = h.$ $AB^2 + OA^2 = OB^2$ $100^2 + h^2 = 3h^2$ $2h^2 = 100^2 = 10000$ $h^2 = 5000$ $h = 50\sqrt{2}$
(c) (iii)	$\tan \angle AOB = \frac{100}{OA} = \frac{100}{50\sqrt{2}} = \sqrt{2}$ $\therefore \angle AOB = 54.7°.$ The bearing of B from O is $180^0 - 54.7^0 = 125.3^0$

Question 4

(a) (i)

Step 1: Show true for $n=1$

$LHS=1$

$RHS=\frac{1}{6}(1+1)(1+2)=1$

True for n=1

Step 2: Assume true for say k

i.e. $1+3+6+...+\frac{1}{2}k(k+1)=\frac{1}{6}k(k+1)(k+2)$

Show true for k+1

i.e. $1+3+6+...+\frac{1}{2}k(k+1)+\frac{1}{2}(k+1)(k+2)=\frac{1}{6}(k+1)(k+2)(k+3)$

$LHS=1+3+6+...+\frac{1}{2}k(k+1)+\frac{1}{2}(k+1)(k+2)$

$=\frac{1}{6}k(k+1)(k+2)+\frac{1}{2}(k+1)(k+2)$

$=\frac{1}{6}(k+1)(k(k+2)+3(k+2))$

$=\frac{1}{6}(k+1)(k+2)(k+3)$

$=RHS$

Step 3: Since true for n=1 (Step 1) then true for n=2 (Step 2)

Since true for n=2 now true for n=3 and so on. Hence by

mathematical induction it is true for all positive integers n.

(b)

Let $f(r)=(1+r)[(1+r)^{24}-1]-50r$

$=(1+r)^{25}-(1+r)-50r$

$=(1+r)^{25}-1-r-50r$

$=(1+r)^{25}-51r-1$

$f'(r)=25(1+r)^{24}-51$

Let $r_1=0.06$

$r_2=r_1-\frac{f(r_1)}{f'(r_1)}$

$=0.06-\frac{f(0.06)}{f'(0.06)}$

$=0.06-\frac{(1.06)^{25}-51\times1.06-1}{25(1.06)^{24}-51}$

≈ 0.055

(c)(i)

(a) Sum of the roots

$\sqrt{k}-\sqrt{k}+\alpha=-p,$

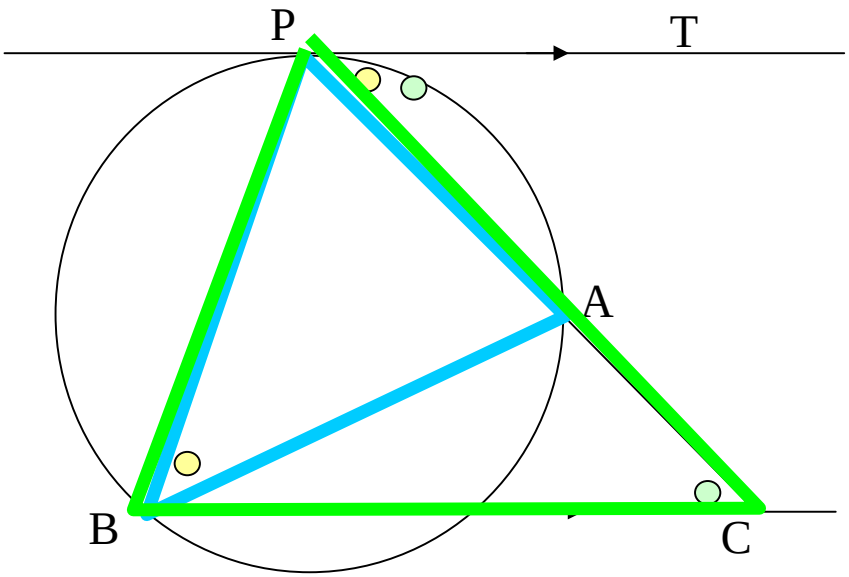
$\therefore \alpha=-p$

or $\alpha+p=0$

(c)(ii)	(b) Product of the roots: $\sqrt{k}\times\left(-\sqrt{k}\right)\times\alpha=-r,$ $-k\alpha=-r,$ $k\alpha=r$
(c)(iii)	(b) Product of the roots: $\sqrt{k}\times\left(-\sqrt{k}\right)\times\alpha=-r,$ $-k\alpha=-r,$ $k\alpha=r$
(d)	Since SHM use $v^2=n^2(a^2-x^2)$ Given Period = 4π *then* $4\pi=\frac{2\pi}{n}$ $\therefore n=\frac{1}{2}$ At O $x=0$. Also given amplitude = 3 $\therefore \quad v^2=n^2(a^2-x^2)$ $=\left(\frac{1}{2}\right)^2(3^2-0^2)=\frac{9}{4}$ $\therefore \quad v=\pm\frac{3}{2}$ $\therefore$ it's speed $=\frac{3}{2}cms^{-1}$

Question 5

(a) (i)



$\angle$ TPA=$\angle$ PBA (alternate angle theorem)

$\angle$ TPA=$\angle$ PCB (alternate angles on parallel lines)

$\therefore$ $\angle$ PBA = $\angle$ PCB (both equal $\angle$ TPA)

(a) (ii)

ΔPAB (blue)||| ΔPCB (green) ($\angle$PBA = $\angle$PCB & $\angle$P is in common – AA)

$\frac{PB}{PA} = \frac{PC}{PB}$ (corresponding sides in similar Δ's)

$\therefore PB^2 = PA \times PC$

(b) (i)

$f'(x) = \frac{x+2-x}{(x+2)^2} = \frac{2}{(x+2)^2} > 0$ for all x ≠ -2

(b)(ii)

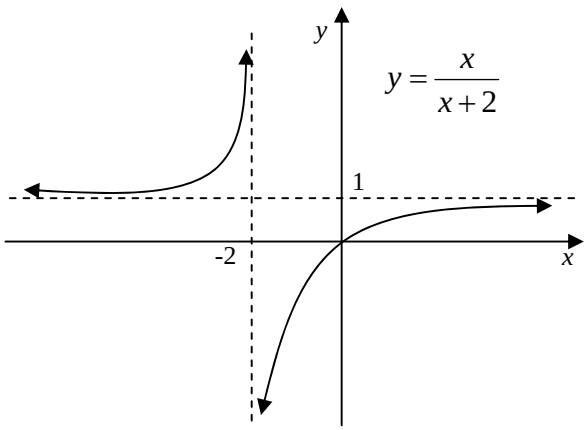
$y = \frac{x}{x+2} = \frac{x+2-2}{x+2} = 1 - \frac{2}{x+2}$

$y - 1 = -\frac{2}{x+2}$

Horizontal asymptote $y - 1 = 0$

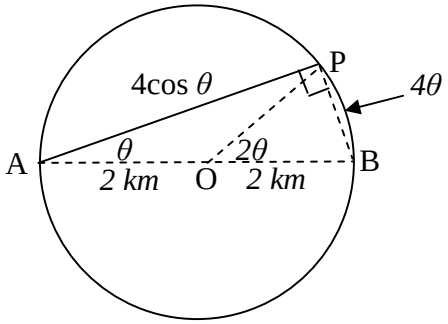
$y = 1$

(b)(iii)



(b)(iv)	Each value of x has a 1-1 corresponding value for y. It would pass the horizontal line test.
(b)(v)	$y = \frac{x}{x+2}$ $y(x+2) = x$ $yx + 2y = x$ $xy - x = 2y$ $x(y-1) = 2y$ $x = \frac{2y}{y-1}$ $\therefore f^{-1}(x) = \frac{2x}{x-1}$
(b)(vi)	$All \ x \neq 1$

Question 6



(a) (i)

$\angle APB = 90^0$ angle in a semi circle

$\Delta APB,\ \cos\theta = \dfrac{AP}{AB} = \dfrac{AP}{4}$

$\therefore AP = 4\cos\theta$

$Arc\ PB = r2\theta = 2\times 2\theta = 4\theta$ (length of an arc)

Now $Speed = \dfrac{\text{distance}}{\text{time}}$ & $Time = \dfrac{\text{distance}}{\text{speed}}$

$$\therefore T = \frac{AP}{s_{ap}} + \frac{PB}{s_{pb}} = \frac{4\cos\theta}{3} + \frac{4\theta}{4} = \frac{4}{3}\cos\theta + \theta$$

$$= \frac{1}{3}(4\cos\theta + 3\theta)$$

(a) (ii)

$$\frac{dT}{d\theta} = \frac{1}{3}(-4\sin\theta + 3) = 0$$

$$-4\sin\theta + 3 = 0$$

$$\sin\theta = \frac{3}{4}$$

$\theta \approx 0.848$ *radians*

(a) (iii)	$\frac{d^2T}{d\theta^2} = \frac{1}{3}(-4\cos\theta)$ $at\ \theta = 0.848$ $\frac{d^2T}{d\theta^2} < 0$ $\therefore$ this time is a maximum since only turning point for $0 \le \theta \le \frac{\pi}{2}$ then test end points At $\theta = 0\ T = \frac{1}{3}(4\cos 0 + 3\times 0) = \frac{4}{3}$ At $\theta = \frac{\pi}{2}\ T = \frac{1}{3}(4\cos\frac{\pi}{2} + 3\times\frac{\pi}{2}) = \frac{\pi}{2}$ Since $\frac{4}{3} < \frac{\pi}{2}$ then T is a minimum at θ=0 i.e. he rows straight across the lake from A to B
(b) (i)	(i) ${}^{13}C_2\ {}^{13}C_4 = 55770$
(b) (ii)	(ii) If all 6 cards are the same suit then combinations: $4\times {}^{13}C_6 = 6864$ If 5 cards are the same suit and one is from one of the other 3 suits with 13 cards each then combinations: $4\times {}^{13}C_5 \times 3\times 13 = 200772$ Total combinations = $6864 + 200772 = 207636$

Question 7

(a) (i)

$$F = Au^3 + \frac{B}{u}$$

$$\frac{dF}{du} = 3Au^2 - \frac{B}{u^2}$$

Stationary points occur when

$$\frac{dF}{du} = 0$$

$$\therefore 3Au^2 - \frac{B}{u^2} = 0$$

$$3Au^2 = \frac{B}{u^2}$$

$$3Au^4 = B$$

$$u^4 = \frac{B}{3A}$$

$$u = \left(\frac{B}{3A}\right)^{\frac{1}{4}} \; km/h$$

Check a minimum

$$\frac{d^2F}{du^2} = 6Au + \frac{2B}{u^3}$$

$$= 6A\left(\frac{B}{3A}\right)^{\frac{1}{4}} + \frac{2B}{\left(\frac{B}{3A}\right)^{\frac{3}{4}}} > 0$$

$\therefore$ a minimum since A and B both >0

$\therefore u = \left(\frac{B}{3A}\right)^{\frac{1}{4}} \; km/h$ is the most efficient speed.

(a) (ii)	Let R be the rate of fuel usage in $litres/km$ $R = \frac{Fuel \ \ (litres/hour)}{Speed \ \ (km/hour)}$ $R = \frac{F}{u}$ $R = \frac{Au^3 + \frac{B}{u}}{u}$ $R = Au^2 + \frac{B}{u^2}$ $\frac{dR}{du} = 2Au - \frac{2B}{u^3}$ $0 = 2Au - \frac{2B}{u^3}$ $2Au = \frac{2B}{u^3}$ $2Au^4 = 2B$ $u^4 = \frac{2B}{2A} = \frac{B}{A}$ $u = \left(\frac{B}{A}\right)^{\frac{1}{4}}$ Check a minimum $\frac{d^2R}{du^2} = 2A + \frac{6B}{u^4} > 0$ when $u = \left(\frac{B}{A}\right)^{\frac{1}{4}}$ Now compare speed to minimise fuel against speed used to maximise distance $\frac{\left(\frac{B}{A}\right)^{\frac{1}{4}}}{\left(\frac{B}{3A}\right)^{\frac{1}{4}}} = \left(\frac{B}{A}\right)^{\frac{1}{4}} \times \left(\frac{3A}{B}\right)^{\frac{1}{4}} = 3^{\frac{1}{4}} \approx 1.32$ The pilot should fly 32% faster.

(b) (i)	*Given* : $x = Vt\cos\theta$ .....(1) $y = Vt\sin\theta - \frac{1}{2}gt^2$ .....(2) $\frac{2V^2}{g} = 1$ .....(3) From (1) $t = \frac{x}{V\cos\theta}$ Sub into (2) $y = V\frac{x}{V\cos\theta}\sin\theta - \frac{1}{2}g\left(\frac{x}{V\cos\theta}\right)^2$ $y = x\tan\theta - \frac{gx^2}{2V^2\cos^2\theta}$ $y = x\tan\theta - \frac{x^2}{\cos^2\theta}$ $y = x\tan\theta - x^2\sec^2\theta$

(b) (ii)	The point T is the intersection of the projectile with the inclined plane The inclined plane has the equation $y = mx + b$ $b = 0$ & m=$\tan\alpha$ $\therefore y = x\tan\alpha$ .....(1) Need to find intersection point T & then use $\cos\alpha = \frac{x}{r}$ Now $y = x\tan\theta - x^2\sec^2\theta$ .....(2) Equating (1) & (2) $x\tan\alpha = x\tan\theta - x^2\sec^2\theta$ $\tan\alpha = \tan\theta - x\sec^2\theta$ $x\sec^2\theta = \tan\theta - \tan\alpha$ $x\sec^2\theta = \frac{\sin\theta}{\cos\theta} - \frac{\sin\alpha}{\cos\alpha}$ $x\sec^2\theta = \frac{\sin\theta\cos\alpha - \cos\theta\sin\alpha}{\cos\theta\cos\alpha}$ $x\sec^2\theta = \frac{\sin(\theta-\alpha)}{\cos\theta\cos\alpha}$ $x = \frac{\sin(\theta-\alpha)}{\cos\theta\cos\alpha\sec^2\theta}$ $x = \frac{\sin(\theta-\alpha)\cos^2\theta}{\cos\theta\cos\alpha}$ $x = \frac{\sin(\theta-\alpha)\cos\theta}{\cos\alpha}$ From (3) $r = \frac{x}{\cos\alpha}$ $\therefore r = \frac{\sin(\theta-\alpha)\cos\theta}{\cos\alpha} \times \frac{1}{\cos\alpha}$ $\therefore r = \frac{\sin(\theta-\alpha)\cos\theta}{\cos^2\alpha}$

b (iii)

$$\frac{dr}{d\theta} = \frac{\sin(\theta-\alpha).(-\sin\theta)+\cos\theta.\cos(\theta-\alpha)}{\cos^2\alpha}$$

$$= \frac{\cos\theta.\cos(\theta-\alpha)-\sin(\theta-\alpha).\sin\theta}{\cos^2\alpha}$$

$$= \frac{\cos(\theta+\theta-\alpha)}{\cos^2\alpha} = \frac{\cos(2\theta-\alpha)}{\cos^2\alpha}$$

Possible maximum when $\frac{dr}{d\theta}=0$

$$\therefore \frac{\cos(2\theta-\alpha)}{\cos^2\alpha}=0$$

$$\cos(2\theta-\alpha)=0$$

$$\therefore 2\theta-\alpha=\frac{\pi}{2} \text{ or } 2\theta=\frac{\pi}{2}+\alpha \text{ or } \theta=\frac{\pi}{4}+\frac{\alpha}{2}$$

$$\frac{d^2r}{d\theta^2}=\frac{-2\sin(2\theta-\alpha)}{\cos^2\alpha} \text{ and at } \theta=\frac{\pi}{4}+\frac{\alpha}{2}$$

$$\frac{d^2r}{d\theta^2} = \frac{-2\sin(2\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)-\alpha)}{\cos^2\alpha} = \frac{-2\sin\frac{\pi}{2}}{\cos^2\alpha} < 0$$

$$\therefore \text{a maximum when } \theta=\frac{\pi}{4}+\frac{\alpha}{2}$$

$$\therefore R = \frac{\sin(\frac{\pi}{4}+\frac{\alpha}{2}-\alpha).\cos(\frac{\pi}{4}+\frac{\alpha}{2})}{\cos^2\alpha}$$

$$R = \frac{\sin(\frac{\pi}{4}-\frac{\alpha}{2}).\cos(\frac{\pi}{4}+\frac{\alpha}{2})}{1-\sin^2\alpha}$$

$$R = \frac{2\sin(\frac{\pi}{4}-\frac{\alpha}{2}).\cos(\frac{\pi}{4}+\frac{\alpha}{2})}{2(1-\sin^2\alpha)}$$

$$R = \frac{\sin(\frac{\pi}{4}-\frac{\alpha}{2}+\frac{\pi}{4}+\frac{\alpha}{2})+\sin(\frac{\pi}{4}-\frac{\alpha}{2}-(\frac{\pi}{4}+\frac{\alpha}{2}))}{2(1-\sin^2\alpha)}$$

using $2\sin A\cos B=\sin(A+B)+\sin(A-B)$

$$R = \frac{\sin\frac{\pi}{2}+\sin(-\alpha)}{2(1-\sin^2\alpha)}$$

$$R = \frac{1-\sin\alpha}{2(1-\sin\alpha)(1-\sin\alpha)}$$

$$R = \frac{1}{2(1-\sin\alpha)}$$

(b) (iv)

The slope of the initial projection is $m_0 = \tan\theta$

This will be a maximum when $\theta = \frac{\alpha}{2} + \frac{\pi}{4}$

$$\therefore m = \tan\left(\frac{\alpha}{2} + \frac{\pi}{4}\right) = \frac{\tan\frac{\alpha}{2} + \tan\frac{\pi}{4}}{1 - \tan\frac{\alpha}{2}\tan\frac{\pi}{4}} = \frac{\tan\frac{\alpha}{2} + 1}{1 - \tan\frac{\alpha}{2}} = \left(\frac{1 + \tan\frac{\alpha}{2}}{1 - \tan\frac{\alpha}{2}}\right)^{**}$$

The slope of the direction on impact is $m_i = \frac{dy}{dx}$

when $x = \frac{\sin(\theta - \alpha)\cos\theta}{\cos\alpha}$ in $y = x\tan\theta - x^2\sec^2\theta$

Need to show $m_o m_i = -1$

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$y = x\tan\theta - x^2\sec^2\theta$

$$\frac{dy}{dx} = \tan\theta - 2x\sec^2\theta = \tan\theta - 2\frac{\sin(\theta - \alpha)\cos\theta}{\cos\alpha}.\frac{1}{\cos^2\theta}$$

$$= \tan\theta - 2\frac{\sin(\theta - \alpha)}{\cos\alpha\cos\theta} = \tan\theta - 2\frac{\sin\theta\cos\alpha - \cos\theta\sin\alpha}{\cos\alpha\cos\theta}$$

$$= \tan\theta - 2(\tan\theta - \tan\alpha) = 2\tan\alpha - \tan\theta$$

$$= 2\left[\frac{2\tan\frac{\alpha}{2}}{1 - \tan^2\frac{\alpha}{2}}\right] - \left(\frac{1 + \tan\frac{\alpha}{2}}{1 - \tan\frac{\alpha}{2}}\right)^{**}$$

$$= \left[\frac{4\tan\frac{\alpha}{2}}{1 - \tan^2\frac{\alpha}{2}}\right] - \left(\frac{1 + \tan\frac{\alpha}{2}}{1 - \tan\frac{\alpha}{2}}\right) = \frac{4\tan\frac{\alpha}{2} - \left(1 + \tan\frac{\alpha}{2}\right)^2}{1 - \tan^2\frac{\alpha}{2}}$$

$$= \frac{4\tan\frac{\alpha}{2} - \left(1 + 2\tan\frac{\alpha}{2} + \tan^2\frac{\alpha}{2}\right)}{1 - \tan^2\frac{\alpha}{2}}$$

$$= \frac{4\tan\frac{\alpha}{2} - 1 - 2\tan\frac{\alpha}{2} - \tan^2\frac{\alpha}{2}}{1 - \tan^2\frac{\alpha}{2}} = \frac{-1 + 2\tan\frac{\alpha}{2} - \tan^2\frac{\alpha}{2}}{1 - \tan^2\frac{\alpha}{2}}$$

$$= -\frac{1 - 2\tan\frac{\alpha}{2} + \tan^2\frac{\alpha}{2}}{1 - \tan^2\frac{\alpha}{2}} = -\frac{\left(1 - \tan\frac{\alpha}{2}\right)^2}{1 - \tan^2\frac{\alpha}{2}} = -\frac{1 - \tan\frac{\alpha}{2}}{1 + \tan\frac{\alpha}{2}}$$

Now $= m_o m_i = \frac{1 + \tan\frac{\alpha}{2}}{1 - \tan\frac{\alpha}{2}}.\left[-\frac{1 - \tan\frac{\alpha}{2}}{1 + \tan\frac{\alpha}{2}}\right] = -1$