1999 HSC Full Solutions

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QUESTION 1

(a)

$$\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$$

$$= \int_0^{\sqrt{3}} \frac{dx}{\sqrt{2^2-x^2}}$$

$$= \left[\sin^{-1}\frac{x}{2}\right]_0^{\sqrt{3}}$$

$$= \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\frac{0}{2}$$

$$= \frac{\pi}{3}$$

(b)

$$y = \sin^3 x$$

$$y = (\sin x)^3$$

$$\frac{dy}{dx} = 3\sin^2 x \cos x$$

(c)

$A(-2,7)$ $B(8,-8)$

$2:3$

$$x = \frac{3.-2+2.8}{2+3} \qquad y = \frac{3.7+2.-8}{2+3}$$

$$x = 2 \qquad y = 1$$

$P(2,1)$

(d)

The vertical asymptote of

$y = \frac{4x}{x-3}$ is $x-3=0$ *i.e.* $x=3$

(e)

$$P(x) = x^3 - 4x$$

$$P(-3) = (-3)^3 - 4.(-3)$$

$$= -27+12 = -15 \therefore \text{the remainder } = -15$$

(f)

Let $u = \tan x$

$$\therefore \frac{du}{dx} = \sec^2 x \quad du = \sec^2 x.dx$$

$$At\ x = 0\ u = 0 \qquad At\ x = \frac{\pi}{3},\ u = \sqrt{3}$$

$$\therefore \int_0^{\frac{\pi}{3}} \tan^2 x \sec^2 x.dx = \int_0^{\sqrt{3}} u^2 du$$

$$= \left[\frac{u^3}{3}\right]_0^{\sqrt{3}} = \left(\frac{\sqrt{3}^3}{3}\right) - \left(\frac{0^3}{3}\right) = \sqrt{3}$$

Question 2

(a)

Choose 2 males from 4 - ${}^4C_2$

Choose 3 females from 7 - ${}^7C_3$

${}^4C_2 \times {}^7C_3 = 210$

(b)

$$1\cos\theta + \sqrt{3}\sin\theta = 1 \quad 0 \le \theta \le 2\pi$$

need to put in the form $r\sin\alpha\cos\theta + r\cos\alpha\sin\theta = r\sin(\alpha+\theta)$

$\therefore \quad r\sin\alpha = 1 \quad$ *or* $\quad \sin\alpha = \frac{1}{r} \ldots(1)$

& $\quad r\cos\alpha = \sqrt{3} \quad$ *or* $\quad \cos\alpha = \frac{\sqrt{3}}{r} \ldots(2)$

Use $\sin^2\alpha + \cos^2\alpha = 1$

$$\frac{1}{r^2} + \frac{\left(\sqrt{3}\right)^2}{r^2} = 1$$

$$1 + 3 = r^2$$

$$r = 2$$

sub into (1)

$$2\sin\alpha = 1$$

$$\sin\alpha = \frac{1}{2}$$

$$\alpha = \frac{\pi}{6}$$

$$\therefore \cos\theta + \sqrt{3}\sin\theta = 1$$

becomes $2\sin\frac{\pi}{6}\cos\theta + 2\cos\frac{\pi}{6}\sin\theta = 1$

$\therefore$ *Now solve* $2\sin(\theta + \frac{\pi}{6}) = 1$

$$\sin(\theta + \frac{\pi}{6}) = \frac{1}{2}$$

Since $0 \le \theta \le 2\pi$ then $0 + \frac{\pi}{6} \le \theta + \frac{\pi}{6} \le 2\pi + \frac{\pi}{6}$

$$\therefore (\theta + \frac{\pi}{6}) = \frac{\pi}{6}, \pi - \frac{\pi}{6} 2\pi + \frac{\pi}{6}.$$

$$\therefore \theta = 0, \frac{2\pi}{3} \text{ or } 2\pi$$

(c) (i)

x>0

(c) (ii)

$f'(x) = 1 + \frac{1}{x}$ where x>0

$\therefore f'(x)>0$ for all x in the functions domain

$\therefore y = f(x)$ is increasing

(c)(iii)	$f(0.5) = 0.5 + \ln 0.5$ $\approx 0.6 - 0.69 < 0$ $f(1) = 1 + \ln 1$ $= 1 > 0$ $\therefore$ The curve crosses the x axis between x=0.5 and 1.
(c)(iv)	$x_2 = x_1 - \dfrac{f(x)}{f'(x)}$ $x_1 = 0.5$ $x_2 = 0.5 - \dfrac{f(0.5)}{f'(0.5)}$ $\approx 0.5 - \dfrac{0.5 - 0.69}{1 + \dfrac{1}{0.5}}$ ≈ 0.56 to 3 dec. pl.

Question 3

(a)

$y = 3\sin x$

$y^2 = 9\sin^2 x dx$

$V = \pi\int_a^b y^2 dx$

$V = \pi\int_0^{\frac{\pi}{2}} 9\sin^2 x dx$

$V = 9\pi\int_0^{\frac{\pi}{2}} \sin^2 x dx$

$V = 9\pi\int_0^{\frac{\pi}{2}} \frac{1-\cos 2x}{2} dx$

$V = \frac{9\pi}{2}\int_0^{\frac{\pi}{2}} 1-\cos 2x.dx$

$V = \frac{9\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$

$V = \frac{9\pi}{2}\left(\left(\frac{\pi}{2} - \frac{1}{2}\sin\pi\right) - (0-0)\right)$

$V = \frac{9\pi^2}{4} units^3$

(b)

${}^7C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^5 \approx 0.234$

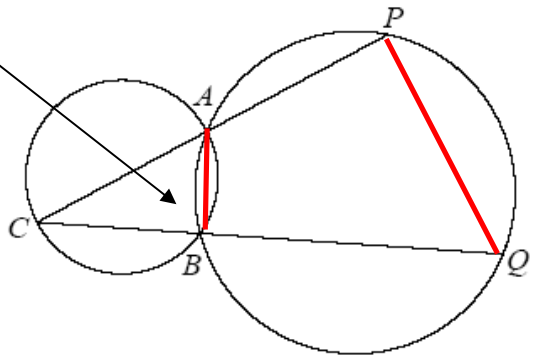
(c)

Construct lines AB &PQ

In circle ABC, $\angle ABC = 90^0$ (angle in a semi-circle)

$\angle CPQ = \angle ABC$ (ext $\angle$ in a cyclic quad = opp. int. $\angle$)

$\therefore \angle CPQ = 90^0$



(d)(i)	$= A(2\sin x + \cos x) + B(2\cos x - \sin x)$ $= 2A\sin x + A\cos x + 2B\cos x - B\sin x$ $= 2A\sin x - B\sin x + A\cos x + 2B\cos x$ $= (2A - B)\sin x + (A + 2B)\cos x$ Now this is equivalent to $\equiv \quad 1\sin x \quad + \quad 8\cos x$ then $2A - B = 1$ .....(1) $A + 2B = 8$ .....(2) $(1) \times 2$ $4A - 2B = 2$ .....(3) $(2) + (3)$ $5A = 10 \quad \therefore A = 2$ sub into (1) $2 \times 2 - B = 1 \quad \therefore B = 3$
(d) (ii)	$\int \frac{\sin x + 8\cos x}{2\sin x + \cos x} dx$ $= \int \frac{2(2\sin x + \cos x) + 3(2\cos x - \sin x)}{2\sin x + \cos x} dx$ from part (i) $= \int 2 + \frac{3(2\cos x - \sin x)}{2\sin x + \cos x} dx$ $= 2x + 3\int \frac{2\cos x - \sin x}{2\sin x + \cos x} dx \qquad use: \int \frac{f'(x)}{f(x)} dx = \ln f(x) + C$ $= 2x + 3\ln(2\sin x + \cos x) + C$

Question 4

(a) (i)

$$\sum_{k=2}^{5}(-1)^k k$$
$$= (-1)^2 2 + (-1)^3 3 + (-1)^4 4 + (-1)^5 5$$
$$= 2 - 3 + 4 - 5$$
$$= -2$$

(b) (i)

(i) $\frac{dy}{dx} = \frac{dy/dp}{dx/dp} = \frac{2p}{2} = p$

Equation of the tangent

$y - p^2 = p(x - 2p)$

$y - p^2 = px - 2p^2$

$y = px - p^2$

(b) (ii)

Equation of the normal:

$y - p^2 = -\frac{1}{p}(x - 2p)$

$py - p^3 = -x + 2p$

$x + py = 2p + p^3$

From the equation of the normal, let x = 0

$y = \frac{2p + p^3}{p} = 2 + p^2,$

$\therefore B(0, 2 + p^2)$

(b) (iii)

From the equation of the tangent, let

$y = 0,$

$px - p^2 = 0$

$\therefore x = p$

$\therefore A(p, 0)$

The midpoint of AB has coordinates

$x = \frac{p}{2}$ (1)

$y = 1 + \frac{p^2}{2}$ (2)

(1) gives $p = 2x$

Put to (2) gives $y = 1 + \frac{4x^2}{2} = 1 + 2x^2.$

The locus of C is $y = 1 + 2x^2.$

(c)(i)

$\int_1^2 \frac{dx}{x} = \left[\ln x\right]_1^2$

$= \ln 2 - \ln 1$

$= \ln 2$

(c)(ii)	$\int_1^2 \frac{dx}{x} = \frac{1}{6}(b-a)\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$ $= \frac{1}{6}(2-1)\left(f(1) + 4f\left(\frac{1+2}{2}\right) + f(2)\right)$ $= \frac{1}{6}\left(1 + 4\times\frac{2}{3} + \frac{1}{2}\right)$ $= \frac{1}{6}\left(\frac{6+16+3}{6}\right)$ $= \frac{25}{36} \approx 0.694$
(c)(iii)	From parts (i) and (ii) $\ln 2 \approx 0.694$ $\therefore 2 = e^{0.694}$ $\therefore \quad \left(e^{0.694}\right)^{\frac{1}{0.694}} = 2^{\frac{1}{0.694}}$ ≈ 2.713 to 3 dec. pl. using calculator

Question 5

(a)

Step 1: Show true for $n=1$

$LHS=(1+1)=2$

$RHS=2^1\left(2\times1-1\right)=2\times1=2$

True for n=1

Step 2: Assume true for say k

i.e. $(k+1)(k+2)+\ldots+(2k-1)2k=2^k(1\times3\times\ldots\times(2k-1))$

Show true for k+1

i.e. $(k+1+1)(k+1+2)+\ldots+(2(k+1)-1)2(k+1)=2^{k+1}(1\times3\times\ldots\times(2(k+1)-1))$

i.e. $(k+2)(k+3)\times\ldots\times(2k+1)2(k+1)=2^{k+1}(1\times3\times\ldots\times(2k+1))$

$$LHS=(k+2)(k+3)\times\ldots\times(2k+1)2(k+1)$$
$$=(k+2)(k+3)\times\ldots\times2k(2k+1)2(k+1)$$
$$=\frac{(k+1)(k+2)(k+3)\times\ldots\times2k(2k+1)2(k+1)}{(k+1)}$$
$$=2^k(1\times3\times\ldots\times(2k-1))\times\frac{(2k+1)2(k+1)}{(k+1)}$$
$$=2^k(1\times3\times\ldots\times(2k-1))\times2(2k+1)$$
$$=2^{k+1}(1\times3\times\ldots\times(2k-1)(2k+1))$$
$$=2^{k+1}(1\times3\times\ldots\times(2k+1))$$

Step 3: Since true for n=1 (Step 1) then true for n=2 (Step 2)
Since true for n=2 now true for n=3 and so on. Hence by mathematical induction it is true for all positive integers n.

(b) (i)

$$f(x)=e^x-1-x$$
$$f'(x)=e^x-1$$
$$0=e^x-1$$
$$\therefore\quad x=0$$
$$f''(x)=e^x$$
$$f''(0)=e^0=1>0$$
$\therefore @\ x=0$ there is a minimum

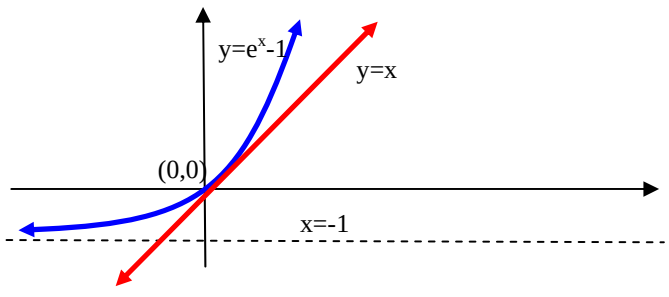
(b) (ii)

$f(0)=e^0-1-0=0$

$\therefore(0,0)$ is the minimum point

Since onle one minimum then $f(x)\geq0$ for all x

(b) (iii)



(b)(iv)	$g: y = e^x - 1$ $g^{-1}: x = e^y - 1$ $x + 1 = e^y$ $e^y = x + 1$ $\ln e^y = \ln(x+1)$ $y \ln e = \ln(x+1)$ $\therefore g^{-1}: y = \ln(x+1)$
(b)(v)	The domain of $g^{-1}(x)$ is the range of g(x) i.e. From the diagram g(x) y>-1 $\therefore$ The domain of $g^{-1}(x)$ is x>-1
(c)(iii)	Since g:$e^x - 1 \geq x$ for all x. See diagram (iii) The blue curve is above the red line. Then its inverse, a reflection in the line y=x will be below y=x over it's domain. $\therefore g^{-1}: \ln(x+1) \leq x$ for $x > -1$

Question 6

(a)(i)

$$\cos^2 3t = \frac{3}{4}$$

$$\cos 3t = \pm\sqrt{\frac{3}{4}}$$

$$\cos 3t = \pm\frac{\sqrt{3}}{2}$$

$$\therefore 3t = \frac{\pi}{6}, \pi - \frac{\pi}{6}, \ldots$$

$$\therefore t = \frac{\pi}{18}, \frac{\pi}{3} - \frac{\pi}{18}, \ldots$$

$\therefore$ first at $x = \frac{3}{4}$ when $t = \frac{\pi}{18}$ seconds

(a)(ii)

$$x = \cos^2 3t$$

$$\frac{dx}{dt} = 2\cos 3t.(-3\sin 3t)$$

$$= -6\cos 3t \sin 3t$$

$$= -3 \times 2\sin 3t \cos 3t$$

$$= -3\sin 6t$$

at $t = \frac{\pi}{18}; \frac{dx}{dt} = -3\sin 6 \times \frac{\pi}{18}$

$$= -3\sin\frac{\pi}{3} = -\frac{3\sqrt{3}}{2} < 0$$

$\therefore$ *velocity* $-ve$ *i.e. to the left*

(a)(iii)

$$x = \cos^2 3t$$

$$\frac{dx}{dt} = 2\cos 3t.(-3\sin 3t)$$

$$= -6\cos 3t \sin 3t$$

$$= -3 \times 2\sin 3t \cos 3t$$

$$= -3\sin 6t$$

$$\frac{d^2x}{dt^2} = -18\cos 6t$$

$$= -18(2\cos^2 3t - 1) \qquad \text{using } \cos 2A = 2\cos^2 A - 1$$

$$= -18(2x - 1)$$

(a)(iv)

$$\frac{d^2x}{dt^2} = -18(2x - 1)$$

$$= -36(x - \frac{1}{2})$$

This is in the form $\frac{d^2x}{dt^2} = -n^2(x - b)$

i.e. SHM

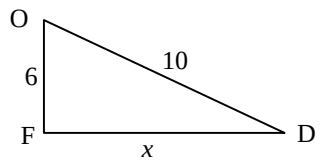
(a)(v)

$-n^2 = 36$

$\therefore n = 6$

$\therefore Period = \frac{2\pi}{n} = \frac{2\pi}{6} = \frac{\pi}{3}$

(b) (i)



$OB = OD = 10cm$ (equal radii of a sphere)

By Pythagoras' Theorem

$FD = 8\ cm$

(b) (ii)

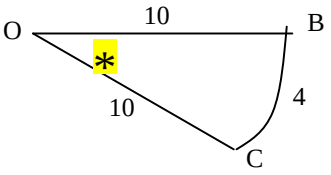
$\angle DFE = \angle BOC$

$Now\ \ l = r\theta$

$\therefore 4 = 10\angle BOC$

$\therefore \angle BOC = 0.4$

$\therefore \angle DFE = 0.4$



(b) (iii)

$Consider\ \ \Delta FED$

$ED^2 = 8^2 + 8^2 - 2\times 8\times 8\times \cos 0.4$

$ED^2 = 128 - 128\cos 0.4$

$Consider\ \ \Delta OED$

$ED^2 = 10^2 + 10^2 - 2\times 10\times 10\times \cos\angle FED$

$ED^2 = 200 - 200\cos\angle FED$

$\therefore 128 - 128\cos 0.4 = 200 - 200\cos\angle FED$

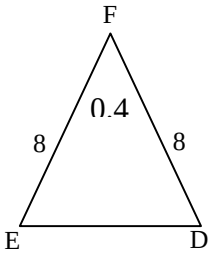
$200\cos\angle FED = 200 - 128 + 128\cos 0.4$

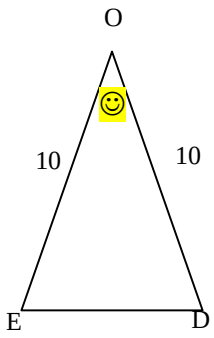
$200\cos\angle FED = 72 + 128\cos 0.4$

$\cos\angle FED = \frac{72 + 128\cos 0.4}{200}$

$\angle FED = \cos^{-1}\left(\frac{72 + 128\cos 0.4}{200}\right)$

$\approx 0.319\ radians$





Question 7

(a) (i)

Given $\ddot{x} = 0$ and $\ddot{y} = -10$

Initial conditions $\dot{x} = 30\cos 5^0$; $\dot{y} = -30\sin 5^0$

Horizontal Motion	Vertical Motion
$\ddot{x} = 0$	$\ddot{y} = -10$
$\dot{x} = \int 0dt = C = 30\cos 5^0$	$\dot{y} = \int -10dt = -10t + C_1 = -10t - 30\sin 5^0$
$x = \int 30\cos 5^0 \, dt$	$y = \int -10t - 30\sin 5^0 dt$
$x = 30t\cos 5^0$	$y = -5t^2 - 30t\sin 5^0$

Note the constants of integration = 0 as the origin is the starting point.

(a) (ii)

The ball strikes the ground at $y = -2$

$\therefore$ Use $y = -5t^2 - 30t\sin 5^0$

i.e. $-2 = -5t^2 - 30t\sin 5^0$

$0 = 5t^2 + 30t\sin 5^0 - 2$

$$\therefore t = \frac{-30\sin 5^0 \pm \sqrt{(30t\sin 5^0)^2 - 4\times 5\times -2}}{2\times 5}$$

$t \approx 0.423$ (ignoring -ve time of course!)

(a) (iii)

If θ is the angle it strikes the ground then

$\tan\theta = \left|\frac{\dot{y}}{\dot{x}}\right|$ when $t \approx 0.423$

$$\tan\theta = \left|\frac{-30\sin 5^0 - 10t}{30\cos 5^0}\right|$$

$$\tan\theta = \left|\frac{-30\sin 5^0 - 10\times 0.423}{30\cos 5^0}\right|$$

$\tan\theta = 0.299$

$\theta = 13^0$ or 167^0

(b)

In the expansion of $(1-x)^n\left(1+\frac{1}{x}\right)^n$

$\binom{n}{2}\binom{n}{0}-\binom{n}{3}\binom{n}{1}+\ldots+(-1)^n\binom{n}{n}\binom{n}{n-2}$

are the coefficients of

$\binom{n}{2}(-x)^2\binom{n}{0}\left(\frac{1}{x}\right)^0+\binom{n}{3}(-x)^3\binom{n}{1}\left(\frac{1}{x}\right)^1+\ldots+\binom{n}{n}(-x)^n\binom{n}{n-2}\left(\frac{1}{x}\right)^{n-2}$

i.e. the coefficient of x^2

$(1-x)^n\left(1+\frac{1}{x}\right)^n=\left((1-x)\left(1+\frac{1}{x}\right)\right)^n=\left(1+\frac{1}{x}-x-1\right)^n=\left(\frac{1}{x}-x\right)^n$

The term containing

x^2 in $\left(\frac{1}{x}-x\right)^n$ is $\binom{n}{r}\left(\frac{1}{x}\right)^{n-r}(-x)^r=\binom{n}{r}(-1)^r x^{-n+r+r}$,

where $-n+r+r=2$

$2r=n+2$

$r=\frac{n+2}{2}$

$\binom{n}{2}\binom{n}{0}-\binom{n}{3}\binom{n}{1}+\ldots+(-1)^n\binom{n}{n}\binom{n}{n-2}=\binom{n}{\frac{n+2}{2}}(-1)^{\frac{n+2}{2}}$

But as $\frac{n+2}{2}$ must be an integer, the formula is true only if n is even.

If n is odd, $\binom{n}{2}\binom{n}{0}-\binom{n}{3}\binom{n}{1}+\ldots+(-1)^n\binom{n}{n}\binom{n}{n-2}=0$ (there is no x^2 term).