1998 HIGHER SCHOOL CERTIFICATE SOLUTIONS

3 UNIT (ADDITIONAL) AND 3/4 UNIT (COMMON) MATHEMATICS

QUESTION 1

(a) $\dfrac{d}{dx}\left(2x\tan^{-1}x\right) = \tan^{-1}x\times 2 + \dfrac{2x}{1+x^2}$

$= 2\tan^{-1}x + \dfrac{2x}{1+x^2}.$

(b) Gradient of $3y = 2x+8$ is $\dfrac{2}{3}$.

Gradient of $y = 5x-9$ is 5.

Let θ be the acute angle between the lines.

$$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

$$= \left|\frac{\frac{2}{3}-5}{1+\frac{2}{3}\times 5}\right|$$

$$= 1$$

$\therefore \quad \theta = 45°.$

(c) $\lim\limits_{x\to 0}\dfrac{\sin 3x}{5x} = \dfrac{3}{5}\times\lim\limits_{3x\to 0}\dfrac{\sin 3x}{3x}$

$= \dfrac{3}{5}\times 1 \qquad \left(\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1\right)$

$= \dfrac{3}{5}.$

(d) $\log_2 14 = \log_2(7\times 2)$

$= \log_2 7 + \log_2 2$

$\doteqdot 2{\cdot}807 + 1$

$= 3{\cdot}807$ (to 3 decimal places).

(e) If $ax^3+bx^2+cx+d = 0$, $\alpha\beta\gamma = -\dfrac{d}{a}$,

$\therefore$ for $2x^3-14x-1 = 0$, $\alpha\beta\gamma = -\dfrac{(-1)}{2} = \dfrac{1}{2}.$

(f) $\displaystyle\int_0^{\frac{\pi}{3}} \sin^2 x\, dx$

$= \displaystyle\int_0^{\frac{\pi}{3}} \frac{1-\cos 2x}{2}\, dx \quad \left(\text{using } \cos 2x = 1-2\sin^2 x\right)$

$= \left[\dfrac{x}{2} - \dfrac{\sin 2x}{4}\right]_0^{\frac{\pi}{3}}$

$= \left(\dfrac{x}{6} - \dfrac{\sin\frac{2\pi}{3}}{4}\right) - (0-\sin 0)$

$= \dfrac{\pi}{6} - \dfrac{\sqrt{3}}{8} - 0$

$= \dfrac{4\pi-3\sqrt{3}}{24}.$

QUESTION 2

(a)
$$\begin{array}{r} x^2-2 \\ x^2+1\overline{)\,x^4-x^2+1} \\ \underline{x^4+x^2} \\ -2x^2+1 \\ \underline{-2x^2-2} \\ 3 \end{array}$$

$\therefore\ Q(x) = x^2-2$ and $R(x) = 3.$

(b) $\$P$, invested for n years at $r\%$ per annum, compounded annually, grows to $\$P(1+r\%)^n$.

$\therefore$ 1st contribution grows to

$\$500(1+8\%)^{40} = \$500(1{\cdot}08)^{40}$

2nd contribution grows to $\$500(1{\cdot}08)^{39}$

$\vdots$

40th contribution grows to $\$500(1{\cdot}08)^1$.

$\therefore$ Amount in fund on 65th birthday

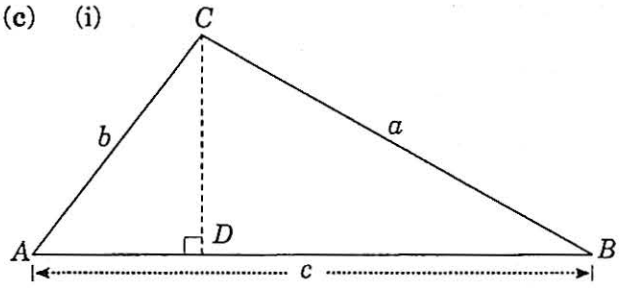
$= \$500\left[(1{\cdot}08)^{40}+(1{\cdot}08)^{39}+(1{\cdot}08)^{38}+\cdots+(1{\cdot}08)\right]$

$= \$500\times a\left(\dfrac{r^n-1}{r-1}\right)$ where $a = 1{\cdot}08$, $r = 1{\cdot}08$, $n = 40$

$= \$500\times 1{\cdot}08\left[\dfrac{(1{\cdot}08)^{40}-1}{1{\cdot}08-1}\right]$

$= \$139\,890{\cdot}52.$

(c) (i)



In ΔCBD, $\sin B = \dfrac{CD}{a}$

$\therefore \quad CD = a\sin B.$ —①

In ΔCAD, $\sin A = \dfrac{CD}{b}$

$\therefore \quad CD = b\sin A.$ —②

$\therefore$ From ① and ②,

$a\sin B = b\sin A.$ —③

(ii) In ΔCBD, $\cos B = \dfrac{DB}{a}$

$\therefore \quad DB = a\cos B.$

In ΔACD, $\cos A = \dfrac{AD}{b}$

$\therefore \quad AD = b\cos A.$

$\therefore \quad AD + DB = b\cos A + a\cos B.$

That is, $c = a\cos B + b\cos A.$ —④

(iii) $c^2 = (a\cos B + b\cos A)^2$ from ④

$= a^2\cos^2 B + 2ab\cos B\cos A + b^2\cos^2 A$

and $c^2 = 4ab\cos A\cos B$ (given)

$\therefore \; a^2\cos^2 B + 2ab\cos B\cos A + b^2\cos^2 A = 4ab\cos A\cos B$

$\therefore \; a^2\cos^2 B - 2ab\cos A\cos B + b^2\cos^2 A = 0$

$\therefore \; (a\cos B - b\cos A)^2 = 0$

$\therefore \quad a\cos B = b\cos A.$ —⑤

Also, $a\sin B = b\sin A$ from ③ —⑥

⑥ ÷ ⑤ $\rightarrow \quad \tan B = \tan A$

$\therefore \quad B = A$ (since $A, B < 180°$)

$\therefore \quad b = a$ (sides opposite equal $\angle$s in ΔACB).

QUESTION 3

(a) When $n = 1$, $4^n + 14 = 4^1 + 14$

$= 18$

$= 6\times 3.$

$\therefore$ Result is true for $n = 1$.

Let k be a value of n for which the result is true.

Then $4^k + 14 = 6M$, where M is an integer. —①

Consider when $n = k+1$,

then $4^{k+1} + 14 = 4^k \times 4 + 14$

$= (6M - 14)\times 4 + 14$ from ①

$= 6\times 4M - 4\times 14 + 14$

$= 6\times 4M - 42$

$= 6(4M - 7).$

That is, it is a multiple of 6 and so the result is true for $n = k+1$.

Hence, since the result is true when $n = k = 1$, it is also true for $n = k = 1+1 = 2$ and by mathematical induction the result will be true for all positive integral values of $n \geq 1$.

(b) (i)

$\sin 4t + \sqrt{3}\cos 4t \equiv R\sin(4t+\alpha)$

$\equiv R(\sin 4t\cos\alpha + \cos 4t\sin\alpha)$

$\equiv R\sin 4t\cos\alpha + R\cos 4t\sin\alpha$

$\therefore \quad R\cos\alpha = 1,$ —①

and $R\sin\alpha = \sqrt{3}.$ —②

② ÷ ① $\rightarrow \quad \tan\alpha = \sqrt{3}$

$\therefore \; \alpha = \dfrac{\pi}{3}, \dfrac{4\pi}{3},$ etc.

Choose $\alpha = \dfrac{\pi}{3}$:

$①^2 + ②^2 \rightarrow \quad R^2\left(\cos^2\alpha + \sin^2\alpha\right) = 4$

$\therefore \quad R^2 = 4$

$\therefore \quad R = \pm 2.$

N.B. $R = 2$ since $\alpha = \dfrac{\pi}{3}$ implies $R > 0$.

Hence $\sin 4t + \sqrt{3}\cos 4t \equiv 2\sin\left(4t + \dfrac{\pi}{3}\right).$ —③

{Other answers are possible, for example: $-2\sin\left(4t + \dfrac{4\pi}{3}\right).$}

(ii) $\sin 4t + \sqrt{3}\cos 4t = 0$

$\therefore \quad 2\sin\left(4t + \dfrac{\pi}{3}\right) = 0$

$\therefore \; 4t + \dfrac{\pi}{3} = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$

$4t = \ldots, -2\pi - \dfrac{\pi}{3}, -\pi - \dfrac{\pi}{3}, 0 - \dfrac{\pi}{3}, \pi - \dfrac{\pi}{3}, 2\pi - \dfrac{\pi}{3}, \ldots$

$= n\pi - \dfrac{\pi}{3}$ for n any integer.

$\therefore \quad t = \dfrac{1}{4}\left(n\pi - \dfrac{\pi}{3}\right)$

$= \dfrac{\pi}{12}(3n - 1)$ for n any integer.

(c) (i) $x = 1 + \sin 4t + \sqrt{3}\cos 4t$

$= 1 + 2\sin\left(4t + \dfrac{\pi}{3}\right)$ from ③. —④

This is of the form $x = b + a\sin(nt + \alpha)$ and hence represents SHM about $x = b = 1$.

Also, $x = 1+2\sin\left(4t+\frac{\pi}{3}\right)$

$= 1+2\cos\left[\frac{\pi}{2}-\left(4t+\frac{\pi}{3}\right)\right]$

$= 1+2\cos\left(\frac{\pi}{6}-4t\right)$

$= 1+2\cos\left(4t-\frac{\pi}{6}\right)$ (since cos is an even function).

This is of the form $x = b+a\cos(nt+\alpha)$ which is the form stated in the syllabus to represent SHM about $x = b = 1$.

OR $x = 1+2\sin\left(4t+\frac{\pi}{3}\right)$ —⑤

$\therefore \quad \dot{x} = 8\cos\left(4t+\frac{\pi}{3}\right)$

$\therefore \quad \ddot{x} = -32\sin\left(4t+\frac{\pi}{3}\right)$

$= -16\times 2\sin\left(4t+\frac{\pi}{3}\right)$ —⑥

$= -4^2(x-1)$ from ⑤.

This is of the form $\ddot{x} = -n^2(x-b)$, and hence represents SHM about the point $x = b = 1$.

(ii) Amplitude = 2.

(iii) Particle is at maximum speed when $\ddot{x} = 0$.

$\ddot{x} = -32\sin\left(4t+\frac{\pi}{3}\right)$ from ⑥

$= 0$ when $\sin\left(4t+\frac{\pi}{3}\right) = 0$.

That is, when $4t+\frac{\pi}{3} = 0, \pi, \ldots$

$4t = \frac{-\pi}{3}, \frac{2\pi}{3}, \ldots$

$t = \frac{-\pi}{12}, \frac{\pi}{6}, \ldots$

$\therefore$ First time particle reaches maximum speed is at $t = \frac{\pi}{6}$.

QUESTION 4

(a) (i) $V = \pi\int_0^h x^2\,dy$ where $x^2+(y-15)^2 = 15^2$

$= \pi\int_0^h \left[15^2-(y-15)^2\right]dy$

$= \pi\int_0^h \left(30y-y^2\right)dy$

$= \pi\left[15y^2-\frac{y^3}{3}\right]_0^h$

$= \pi\left(15h^2-\frac{h^3}{3}\right)-0$

$= 15\pi h^2-\frac{\pi h^3}{3}$ as required.

(ii) $V = 15\pi h^2-\frac{\pi h^3}{3}$

$\frac{dV}{dh} = 30\pi h-\pi h^2$

$\frac{dV}{dt} = 3.$

Find $\frac{dh}{dt}$ when $h = 6$.

$\frac{dV}{dt} = \frac{dV}{dh}\times\frac{dh}{dt}$

$\therefore \quad 3 = \left(30\pi h-\pi h^2\right)\frac{dh}{dt}$

$\therefore \quad \frac{dh}{dt} = \frac{3}{30\pi h-\pi h^2}.$

When $h = 6$, $\frac{dh}{dt} = \frac{3}{30\pi\times 6-\pi\times 6^2}$

$= \frac{1}{48\pi}.$

$\therefore$ Water is rising at a rate of $\frac{1}{48\pi}$ cm s^{-1} when the water level is 6 cm.

(b) (i) $f(x) = 1+\frac{3}{x-2}, \quad x>2.$

Let $y = f(x)$.

Then $y = 1+\frac{3}{x-2}, \quad x>2.$

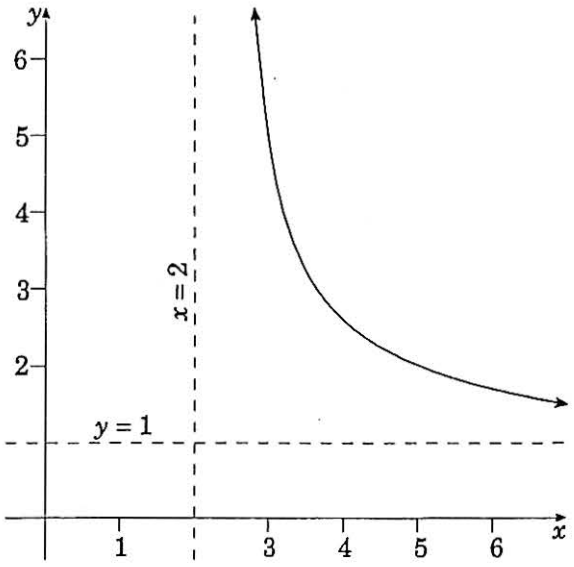
$\therefore \quad y-1 = \frac{3}{x-2}, \quad x>2.$

This is one branch of a rectangular hyperbola of the form

$$Y = \frac{3}{X}$$

translated 2 units to the right and 1 unit up. Hence the horizontal asymptote is $y = 1$ and the vertical asymptote is $x = 2$.

Sketch of $y = f(x)$ for $x > 2$.



(ii) The inverse function is obtained from

$x = 1+\frac{3}{y-2}$, where $y>2$, $x>1$.

$$x-1=\frac{3}{y-2}$$

$$y-2=\frac{3}{x-1}$$

$$y=2+\frac{3}{x-1}$$

$$\therefore \quad f^{-1}(x)=2+\frac{3}{x-1},\ x>1.$$

(iii) The range of $f(x)$ is $y>1$ (see sketch) so the domain of $f^{-1}(x)$ is $x>1$.

(c) (i) If an interval (BC) subtends two equal angles at two points (D, E) on the same side of it, then the endpoints of the interval and the two points are concyclic.

(ii) Exterior $\angle AED$ of cyclic quadrilateral $EDCB$ equals the interior opposite $\angle DCB$.

QUESTION 5

(a) $8+27+\cdots+n^3=2^3+3^3+\cdots+n^3$

$$=\sum_{r=2}^{n} r^3.$$

(b) (i) $P(x)=4x^3+2x^2+1$

$P(0)=1$

$P(1)=7$

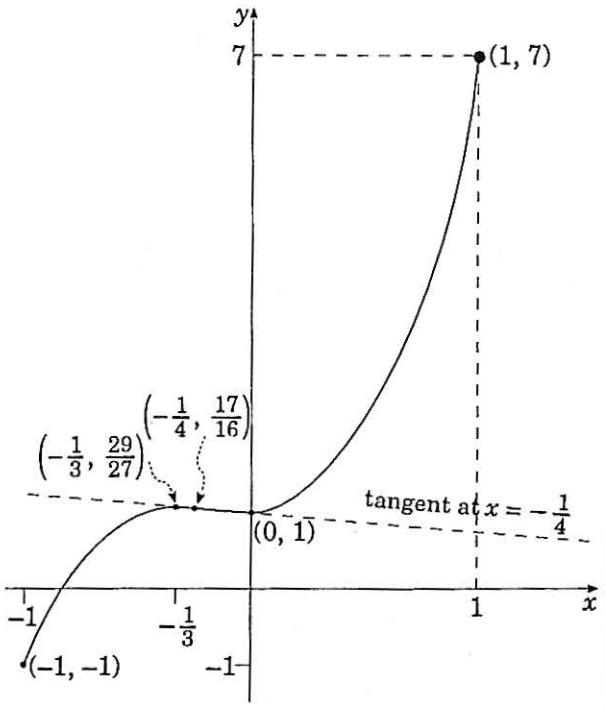
$P(-1)=-1$

$P'(x)=12x^2+4x$

$=4x(3x+1)$

$=0$ when $x=0$ or $-\frac{1}{3}$.

$\therefore$ $(0, 1)$ and $\left(-\frac{1}{3}, \frac{29}{27}\right)$ are stationary points.



The continuous nature of polynomial functions and the positions of the stationary points determine their nature as turning points.

OR $P''(x)=24x+4$

$P''(0)=4>0$

$\therefore$ $(0, 1)$ is a minimum turning point.

$P''\left(-\frac{1}{3}\right)=-4<0$

$\therefore$ $\left(-\frac{1}{3}, \frac{29}{27}\right)$ is a maximum turning point.

(ii) $P(x)=4x^3+2x^2+1$

$P'(x)=12x^2+4x.$

Using Newton's method with $x_1=-\frac{1}{4}$, another approximation is

$$x_2=x_1-\frac{P(x_1)}{P'(x_1)}$$

$$=-\frac{1}{4}-\frac{4\times\left(-\frac{1}{4}\right)^3+2\left(-\frac{1}{4}\right)^2+1}{12\left(-\frac{1}{4}\right)^2+4\left(-\frac{1}{4}\right)}$$

$=4.$

(iii) Newton's method is calculating the x intercept of the tangent to this curve at $x=-\frac{1}{4}$ and using this value as the next approximation to the root. Because there is a turning point between $x=-\frac{1}{4}$ and the root, the tangent to the curve at $x=-\frac{1}{4}$ will not cut the x axis closer to the root. (See graph.)

(c) (i) P(4 untagged fish)

$$=\frac{50}{60}\times\frac{49}{59}\times\frac{48}{58}\times\frac{47}{57}$$

$=0{\cdot}4723$ to 4 significant figures.

(ii) P(at least one tagged fish)

$=1-P$(4 untagged fish)

$=1-0{\cdot}4723$

$=0{\cdot}5277$ to 4 significant figures.

(iii) P(no tagged fish for 7 days)

$=(0{\cdot}4723)^7$

$=0{\cdot}005\,24$ to 3 significant figures.

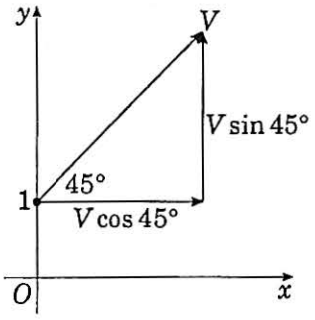
(iv) P(no tagged fish for exactly 2 days)

$={}^7C_2\times(0{\cdot}4723)^2\times(0{\cdot}5277)^5$

$=0{\cdot}192$ to 3 significant figures.

QUESTION 6

(a) (i)



$\ddot{x} = 0$

$\dot{x} = \int 0 \; dt = C_1.$

When $t = 0$, $\dot{x} = V\cos 45° = \dfrac{V}{\sqrt{2}}$

$\therefore \quad C_1 = \dfrac{V}{\sqrt{2}}$

$\therefore \quad \dot{x} = \dfrac{V}{\sqrt{2}}$

$x = \int \dfrac{V}{\sqrt{2}} \; dt$

$= \dfrac{Vt}{\sqrt{2}} + C_2.$

When $t = 0$, $x = 0$

$\therefore \quad C_2 = 0$

$\therefore \quad x = \dfrac{Vt}{\sqrt{2}}$ —①

$\ddot{y} = -g$

$\dot{y} = \int -g \; dt = -gt + C_3.$

When $t = 0$, $\dot{y} = V\sin 45° = \dfrac{V}{\sqrt{2}}$

$\therefore \quad C_3 = \dfrac{V}{\sqrt{2}}$

$\therefore \quad \dot{y} = -gt + \dfrac{V}{\sqrt{2}}$

$y = \int \left(-gt + \dfrac{V}{\sqrt{2}}\right) dt$

$= \dfrac{-gt^2}{2} + \dfrac{Vt}{\sqrt{2}} + C_4.$

When $t = 0$, $y = 1$

$\therefore \quad C_4 = 1$

$\therefore \quad y = \dfrac{-gt^2}{2} + \dfrac{Vt}{\sqrt{2}} + 1.$ —②

From ①, $t = \dfrac{\sqrt{2}x}{V}.$

Substitute in ②:

$y = \dfrac{-g\cancel{2}x^2}{\cancel{2}V^2} + \dfrac{V}{\cancel{\sqrt{2}}} \times \dfrac{\cancel{\sqrt{2}}x}{V} + 1$

$= 1 + x - g\dfrac{x^2}{V^2}.$ —③

(ii) When $x = 9{\cdot}3$, $y = 2{\cdot}3$.

Substitute in ③: $2{\cdot}3 = 1 + 9{\cdot}3 - 9{\cdot}8 \times \dfrac{(9{\cdot}3)^2}{V^2}$

$\therefore \quad \dfrac{9{\cdot}8 \times (9{\cdot}3)^2}{V^2} = 8$

$V^2 = \dfrac{9{\cdot}8 \times (9{\cdot}3)^2}{8}$

$V = 9{\cdot}3\sqrt{\dfrac{9{\cdot}8}{8}}$

$= 10{\cdot}2932 \ldots$

$\doteqdot 10{\cdot}3$

$\therefore$ Initial speed $\doteqdot 10{\cdot}3 \text{ ms}^{-1}$.

(iii) The ball is at ground level when $y = 0$.

Substitute $y = 0$, $V = 10{\cdot}3$, $g = 9{\cdot}8$ in ③:

$0 = 1 + x - \dfrac{9{\cdot}8x^2}{(10{\cdot}3)^2}$

$9{\cdot}8x^2 - (10{\cdot}3)^2 x - (10{\cdot}3)^2 = 0$

$x = \dfrac{(10{\cdot}3)^2 \pm \sqrt{(10{\cdot}3)^4 + 4 \times 9{\cdot}8 \times (10{\cdot}3)^2}}{2 \times 9{\cdot}8}$

$= 11{\cdot}747\,06 \ldots$ (other answer not relevant as it is less than 9·3, by inspection)

$\approx 11{\cdot}7$ m.

The ball lands approximately $(11{\cdot}7 - 9{\cdot}3)$ metres from the net, that is, approximately 2·4 m from the net.

(b) (i) $\dfrac{d}{dx}\left(\frac{1}{2}v^2\right) = 3\left(1 - x^2\right)$

$\frac{1}{2}v^2 = \int 3\left(1 - x^2\right) dx$

$= 3x - x^3 + C.$

When $x = 0$, $v = 4$

$\therefore \quad C = 8$

$\therefore \quad \frac{1}{2}v^2 = 3x - x^3 + 8$

$\therefore \quad v^2 = 6x - 2x^3 + 16$

$= 16 + 6x - 2x^3.$

(ii) Yes.

Consider the graph of $v^2 = 16 + 6x - 2x^3$.

When $x = 0$, $v^2 = 16$.

When $x = 2$, $v^2 = 12$.

When $x = 3$, $v^2 = -20$.

$\therefore$ Curve cuts x axis between $x = 2$ and $x = 3$, (at α say).

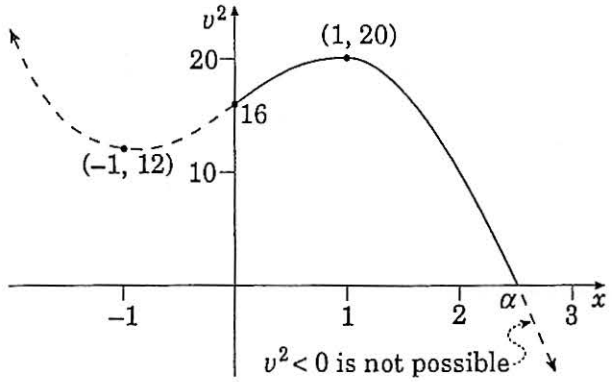
$\dfrac{d}{dx}\left(v^2\right) = -6x^2 + 6$

$= 6\left(1 - x^2\right)$

$= 0$ when $x = \pm 1$.

When $x = 1$, $v^2 = 20$.

When $x = -1$, $v^2 = 12$.



From the graph, it can be seen that the particle stops at $x = \alpha$ (where $v^2 = 0$).

Acceleration $a = 3(1-x^2) < 0$ for $x > 1$, and since $\alpha > 1$, the particle begins to move towards the left. It continues without stopping and will reach the origin since $v^2 \neq 0$ for any other value of x (see graph).

QUESTION 7

(a) (i) $(1+x)^{2n} + (1-x)^{2n}$

$= 1 + {}^{2n}C_1x + {}^{2n}C_2x^2 + {}^{2n}C_3x^3 + \cdots + {}^{2n}C_{2n}x^{2n}$

$\quad + 1 - {}^{2n}C_1x + {}^{2n}C_2x^2 - {}^{2n}C_3x^3 + \cdots + {}^{2n}C_{2n}x^{2n}$

$= 2\left(1 + {}^{2n}C_2x^2 + {}^{2n}C_4x^4 + \cdots + {}^{2n}C_{2n}x^{2n}\right).$

(ii) If $x = 1$ and $n = 10$, (i) becomes

$2^{20} + 0 = 2\left(1 + {}^{20}C_2 + {}^{20}C_4 + \cdots + {}^{20}C_{20}\right)$

$\therefore\ 1 + {}^{20}C_2 + {}^{20}C_4 + \cdots + {}^{20}C_{20} = 2^{19}.$

(b) (i) $y = \sqrt{x}$

$\therefore\quad x = y^2$

$\therefore\quad \dfrac{dx}{dy} = 2y$

$\therefore\quad \displaystyle\int \frac{dx}{\sqrt{x(1-x)}} = \int \frac{2y\,dy}{y\sqrt{1-y^2}}$

$= \displaystyle 2\int \frac{dy}{\sqrt{1-y^2}}$

$= 2\sin^{-1}y + C,$

$= 2\sin^{-1}\sqrt{x} + C, \quad 0 < x < 1.$

(ii) $z = x - \frac{1}{2}$

$\therefore\quad x = z + \frac{1}{2}$

$\therefore\quad \dfrac{dx}{dz} = 1$

$\therefore\quad \displaystyle\int \frac{dx}{\sqrt{x(1-x)}} = \int \frac{dz}{\sqrt{\left(z+\frac{1}{2}\right)\left(\frac{1}{2}-z\right)}}$

$= \displaystyle\int \frac{dz}{\sqrt{\left(\frac{1}{2}\right)^2 - z^2}}$

$= \sin^{-1}\left(\dfrac{z}{\frac{1}{2}}\right) + C,$

$= \sin^{-1}2z + C,$

$= \sin^{-1}(2x-1) + C, \quad 0 < x < 1.$

(iii) Hence $\sin^{-1}(2x-1)$

$= 2\sin^{-1}\sqrt{x} + C, \quad 0 < x < 1.$

To evaluate constant C, put, say, $x = \frac{1}{2}$,

then $\sin^{-1}0 = 2\sin^{-1}\dfrac{1}{\sqrt{2}} + C$

$0 = 2 \times \dfrac{\pi}{4} + C$

$\therefore\quad C = -\dfrac{\pi}{2}.$

Hence $\sin^{-1}(2x-1)$

$= 2\sin^{-1}\sqrt{x} - \dfrac{\pi}{2}, \quad 0 < x < 1.$

Note

The domain $0 < x < 1$ is the domain of the function

$$\frac{1}{\sqrt{x(1-x)}}$$

from which this equation was obtained. However this equation is also true for $x = 0$ and $x = 1$.

(c) $|4x-1| > 2\sqrt{x(1-x)}.$

Using a definition of $|x|$ as $\sqrt{x^2}$, the inequality becomes

$$\sqrt{(4x-1)^2} > 2\sqrt{x(1-x)}. \qquad \text{—①}$$

$\sqrt{x}$ is defined only for $x \geq 0$

$\therefore\quad x(1-x) \geq 0$

$\therefore\quad 0 \leq x \leq 1. \qquad \text{—②}$

Squaring both sides of ①, we get

$(4x-1)^2 > 4x(1-x)$

$\therefore\quad 16x^2 - 8x + 1 > 4x - 4x^2$

$\therefore\quad 20x^2 - 12x + 1 > 0$

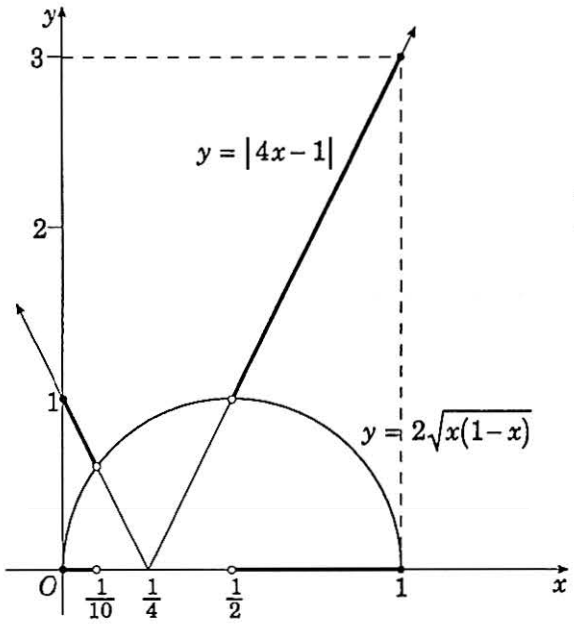
$\therefore\quad (10x-1)(2x-1) > 0$

$\therefore\quad x < \frac{1}{10}$ or $x > \frac{1}{2}. \qquad \text{—③}$

From ② and ③, the solution is

$0 \leq x < \frac{1}{10}$ or $\frac{1}{2} < x \leq 1.$

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The graph confirms the restriction on the domain.

END OF 3 UNIT (ADDITIONAL) AND 3/4 UNIT (COMMON) MATHEMATICS SOLUTIONS