1998 HSC Full Solutions

Question 1☺ Question 2☺ Question 3☺
Question 4☺ Question 5☺ Question 6☺
Question 7☺



QUESTION 1

(a)

$y = 2x\tan^{-1}x \quad (\textit{use the product rule})$

$$\frac{dy}{dx} = \frac{2x}{1+x^2} + 2\tan^{-1}x$$

(b)

$y = 5x - 9 \therefore m_1 = 5$

$$y = \frac{2}{3}x + \frac{8}{3} \therefore m_2 = \frac{2}{3}$$

$$Now \ \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\tan\theta = \left|\frac{5 - \frac{2}{3}}{1 + 5\times\frac{2}{3}}\right|$$

$$= \left|\frac{15-2}{3+10}\right|$$

$$= 1$$

$$\therefore \theta = 45^0$$

(c)

$$\lim_{x\to 0}\frac{\sin 3x}{5x} = \lim_{x\to 0}\frac{3\times\sin 3x}{3\times 5x} = \frac{3}{5}\lim_{x\to 0}\frac{\sin 3x}{3x} = \frac{3}{5}$$

$$Note: \lim_{x\to 0}\frac{\sin 3x}{3x} = 1$$

(d)

$$\log_2 14 = \log_2 2\times 7 = \log_2 2 + \log_2 7$$

$$= 1 + 2.807 = 3.807$$

(e)

$$\alpha\beta\gamma = -\frac{d}{a} = -\frac{-1}{2} = \frac{1}{2}$$

(f)

$$\int_0^{\frac{\pi}{3}} \sin^2 x.dx = \frac{1}{2}\int_0^{\frac{\pi}{3}} (1-\cos 2x).dx$$

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{3} - \frac{1}{2}\sin 2\times\frac{\pi}{3}\right) - \left(0 - \frac{1}{2}\sin 2\times 0\right)\right]$$

$$= \frac{1}{2}\times\frac{\pi}{3} - \frac{1}{2}\times\frac{\sqrt{3}}{2}$$

$$= \frac{\pi}{6} - \frac{\sqrt{3}}{4}$$

Question 2

(a)

$$x^2+1\overline{)x^4+0x^3-x^2+0x+1}\quad x^2-2$$

$$\begin{array}{rr} x^4 & +x^2 \\ \hline & -2x^2 \quad +1 \\ & -2x^2 \quad -2 \\ \hline & 3 \end{array}$$

$\therefore Q(x)=x^2-2$ & $R(x)=3$

(b)

1st investment after 40 years = 500×1.08^{40}

2nd investment after 40 years = 500×1.08^{39}

...

39th investment after 40 years = 500×1.08^{2}

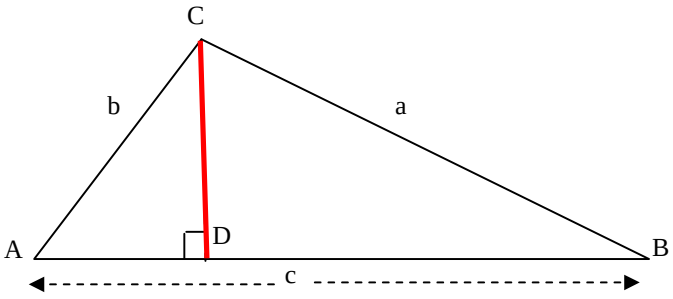
40th investment after 40 years = 500×1.08^{1}

$\therefore Total = 500\times1.08^{1}+500\times1.08^{2}+...+\ 500\times1.08^{39}+500\times1.08^{40}$

This is the sum of a geometric series where $a=500\times1.08$ $r=1.08$ $n=40$

$$\therefore Total=\frac{a(r^n-1)}{r-1}=\frac{500\times1.08(1.08^n-1)}{1.08-1}=\$139890.52$$

(c)



$$In\ \Delta DCB:\ \sin B=\frac{CD}{a}$$

$\therefore CD=a\sin B$

$$In\ \Delta DCA:\ \sin A=\frac{CD}{b}$$

$\therefore CD=b\sin A$

$\therefore a\sin B=b\sin A$

(c)(ii)

$c=AD+DB$

$DB=a\cos B$

$AD=b\cos A$

$\therefore c=a\cos B+b\cos A$

(c)(iii)	$c^2 = \left(a\cos B + b\cos A\right)^2 = a^2\cos^2 B + 2ab\cos A\cos B + b^2\cos^2 A$ *Now* $c^2 = 4ab\cos A\cos B = a^2\cos^2 B + 2ab\cos A\cos B + b^2\cos^2 A$ $\therefore 0 = a^2\cos^2 B - 2ab\cos A\cos B + b^2\cos^2 A$ $\therefore 0 = (a\cos B - b\cos A)^2$ $\therefore 0 = a\cos B - b\cos A$ $a\cos B = b\cos A$ $a^2\cos^2 B = b^2\cos^2 A$ $a^2(1-\sin^2 B) = b^2(1-\sin^2 A)$ $a^2 - a^2\sin^2 B = b^2 - b^2\sin^2 A$ *but* $a\sin B = b\sin A$ *from* (i) $\therefore a^2 = b^2$ $\therefore a = b$ since both are > 0

Question 3

(a) (i)

Step 1: Show true for $n = 1$

$LHS = 4^1 + 14$

$= 18$

$18 \div 6 = 3$

True for n=1

Step 2: Assume true for say k

i.e. $4^k + 14 = 6p$ where p is an integer

or this may be written as $4^k = 6p - 14$

Show true for k+1

i.e. $4^{k+1} + 14 = 6m$ where m is an integer

$LHS = 4^{k+1} + 14$

$= 4^k.4 + 14$

$= (6p - 14)4 + 14$

$= 24p - 56 + 14$

$= 24p - 42$

$= 6(4p - 7)$

$= 6m$

where $m = 4p - 7$

Step 3: Since true for n=1 (Step 1) then true for n=2 (Step 2)

Since true for n=2 now true for n=3 and so on. Hence by

mathematical induction it is true for all positive integers n.

(a) (ii)

(b) (i)

$= R\sin(4t+\alpha)$

$= R\sin 4t\cos\alpha + R\cos 4t\sin\alpha$

$= R\cos\alpha\sin 4t + \quad R\sin\alpha\cos 4t$

$\equiv \quad 1\sin 4t + \quad \sqrt{3}\cos 4t$

$\therefore R\cos\alpha = 1 \qquad \therefore \cos\alpha = \frac{1}{R}$

$\therefore R\sin\alpha = \sqrt{3} \qquad \therefore \sin\alpha = \frac{\sqrt{3}}{R}$

use

$\therefore \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{\sqrt{3}}{R}}{\frac{1}{R}}$

i.e. $\tan\alpha = \sqrt{3} \qquad \alpha = \frac{\pi}{3}$

$\therefore R = \frac{1}{\cos\alpha} = \frac{1}{\cos\frac{\pi}{3}} = \frac{1}{\frac{1}{2}} = 2$

Now

$= R\sin(4t+\alpha)$

$= 2\sin(4t+\frac{\pi}{3})$

(b) (ii)

$\sin 4t + \sqrt{3}\cos 4t = 0$

is equivalent to solving

$2\sin\left(4t+\frac{\pi}{3}\right) = 0$

$\sin\left(4t+\frac{\pi}{3}\right) = 0$

$4t + \frac{\pi}{3} = n\pi$

$4t = n\pi - \frac{\pi}{3}$

$t = \frac{n\pi}{4} - \frac{\pi}{12}$

(c) (i)	$x = 1 + \sin 4t + \sqrt{3}\cos 4t$ $x - 1 = \sin 4t + \sqrt{3}\cos 4t$ $x - 1 = 2\sin(4t + \frac{\pi}{3})$ *from* (b) $\frac{dx}{dt} = 8\cos(4t + \frac{\pi}{3})$ $\frac{d^2x}{dt^2} = -32\sin(4t + \frac{\pi}{3})$ $= -16 \times 2\sin(4t + \frac{\pi}{3})$ $= -16(x - 1)$ $\therefore$ *SHM about* $x = 1$
(c) (ii)	Amplitude = 2
(c) (iii)	$\frac{dx}{dt} = 8\cos(4t + \frac{\pi}{3})$ $\frac{dx}{dt}$ is a maximum when $\cos(4t + \frac{\pi}{3}) = 1$ $\therefore 4t + \frac{\pi}{3} = 0, \pi$ *etc* Since $t \geq 0 : 4t + \frac{\pi}{3} = \pi$ $4t = \pi - \frac{\pi}{3}$ $4t = \frac{2\pi}{3}$ $t = \frac{2\pi}{4 \times 3} = \frac{\pi}{6}$ $\therefore$ first reaches maximum speed when $t = \frac{\pi}{6}$

Question 4

(a) (i)

$V = \pi \int_a^b x^2 dy$

Now $x^2 + (y-15)^2 = 15^2$

$x^2 = 15^2 - (y-15)^2$

$x^2 = 15^2 - (y^2 - 30y + 15^2)$

$x^2 = 15^2 - y^2 + 30y - 15^2$

$x^2 = 30y - y^2$

$\therefore V = \pi \int_0^h 30y - y^2 dy$

$= \pi \left[15y^2 - \frac{y^3}{3} \right]_0^h$

$= \pi \left(15h^2 - \frac{h^3}{3} \right) - (0-0)$

$= 15\pi h^2 - \frac{\pi h^3}{3}$

(a) (ii)

Now $\frac{dV}{dt} = 3$ &

$V = 15\pi h^2 - \frac{\pi h^3}{3}$

$\frac{dV}{dh} = 30\pi h - \pi h^2$

Now $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$= \frac{1}{30\pi h - \pi h^2} \times 3$

when $h = 6$

$= \frac{3}{30\pi 6 - \pi 6^2}$

$= \frac{3}{144\pi} = \frac{1}{48\pi}$

$\therefore$ The rate at which the water is dropping when h=6 is $\frac{1}{48\pi} cms^{-1}$

(b) (i)

$f(x) = 1 + \frac{3}{(x-2)}$ *for* $x > 2$

$y - 1 = \frac{3}{(x-2)}$

Vertical assymptote $x = 2$

Horizontal asymptote $y = 1$

(b) (ii)	$f: y-1=\dfrac{3}{(x-2)}$ $f^{-1}: x-1=\dfrac{3}{(y-2)}$ $: y-2=\dfrac{3}{(x-1)}$ $: y=2+\dfrac{3}{(x-1)}$
(b) (iii)	Domain of $f^{-1}(x)$ is the range of $f(x)$ Range of $f(x): y>1$ $\therefore$ Domain of $f^{-1}(x)$ is $x>1$
(c)(i)	EDBC is a cyclic quadrilateral because BC subtends equal angles from E and from D
(c)(ii)	$\angle$AED = $\angle$ACB because $\angle$AED is an exterior angle to a cyclic quadrilateral and $\angle$ACB is the opposite interior angle.

Question 5

(a)

$= 8 + 27 + ... + n^3$

$= 2^3 + 3^3 + ... + n^3$

$= \sum_{k=2}^{n} k^3$

(b) (i)

$P(x) = 4x^3 + 2x^2 + 1$

$P'(x) = 12x^2 + 4x$

$= 4x(3x+1)$

$P'(x) = 0$

$= 4x(3x+1)$

$\therefore x = 0 \quad \& \; y = 1$

or

$x = -\frac{1}{3} \quad \& \; y = \frac{29}{27}$

$P''(x) = 24x + 4$

$P''(0) = 4 > 0$

$\therefore$ a minimum

$P''\left(-\frac{1}{3}\right) = -4 < 0$

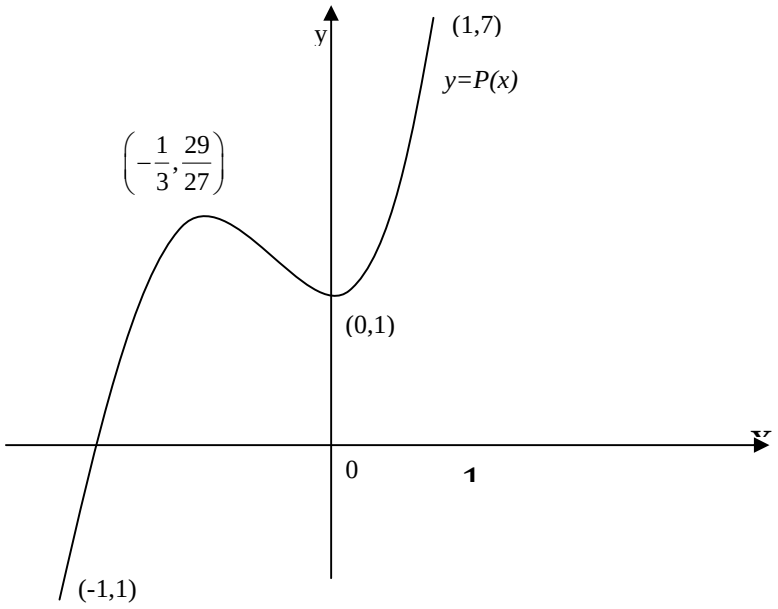
$\therefore$ a maximum

At the end points:

$P(-1) = -1$

$P(1) = 7$

$\therefore$ end points $(-1,-1)$ & $(1,7)$



(b) (ii)

$$x_2 = x_1 - \frac{f(x)}{f'(x)}$$

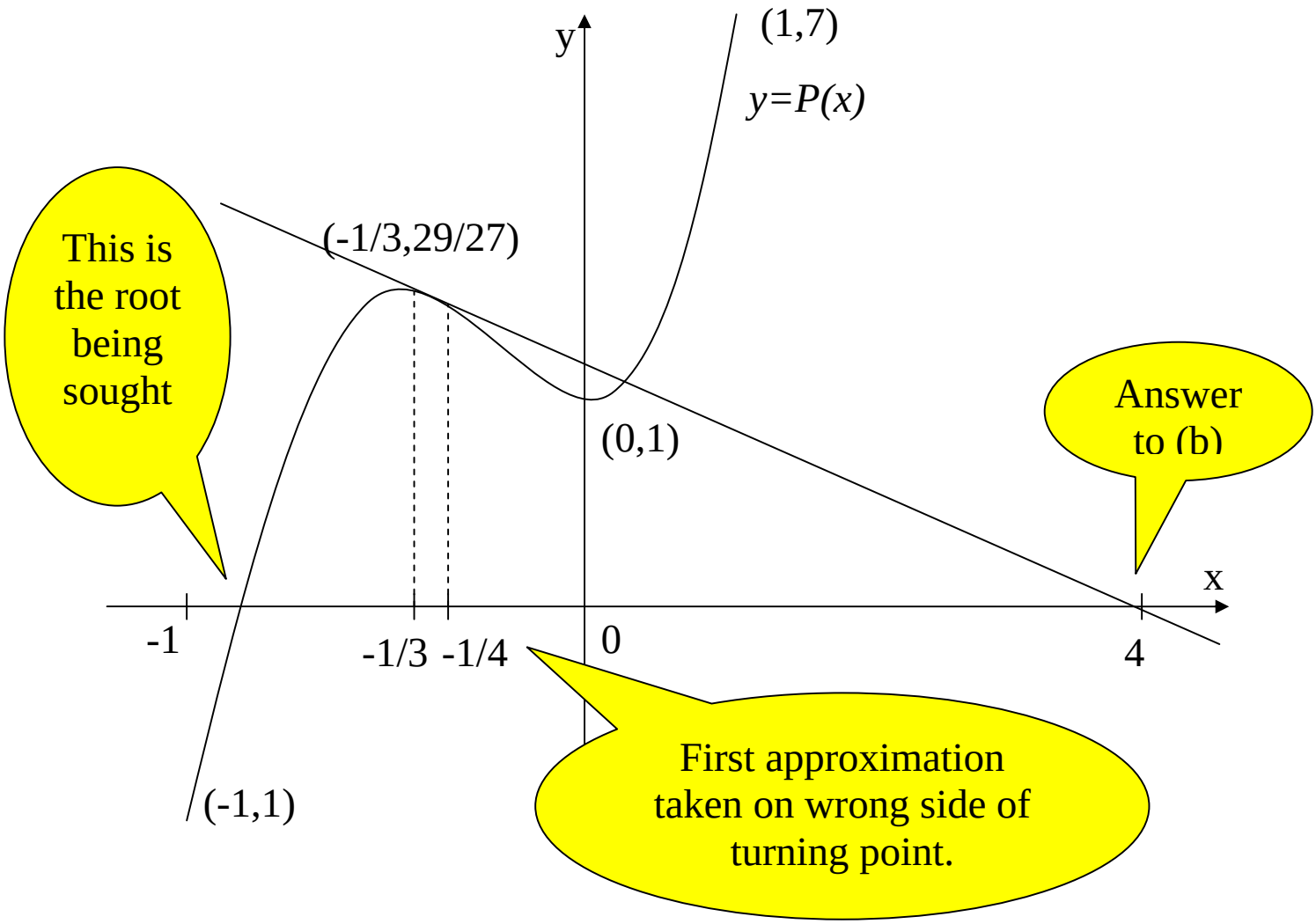
$$x_2 = -\frac{1}{4} - \frac{P\left(-\frac{1}{4}\right)}{P'\left(-\frac{1}{4}\right)}$$

$$x_2 = -\frac{1}{4} - \frac{\frac{17}{16}}{-\frac{4}{16}}$$

$$x_2 = -\frac{1}{4} + \frac{17}{4}$$

$$x_2 = 4$$

(b) (iii)



(c)(i)	$\frac{Number\ of\ combinations\ of\ untagged}{Number\ of\ combinations\ of\ any\ 4\ fish}$ P(4 untagged) = $\frac{{}^{50}C_4}{{}^{60}C_4} \approx 0.47$
(c)(ii)	P(at least 1 tagged) = 1 - P(4 untagged) = 1- $\frac{{}^{50}C_4}{{}^{60}C_4} = 0.53$
(c)(iii)	$P(no\ tagged\ in\ a\ day) = 0.47$ $P(no\ tagged\ in\ 7\ days) = 0.47^7 \approx 0.005$
(c)(iv)	P(no tagged in exactly 2 days) = ${}^7C_2(0.47)^2(0.53)^5 \approx 0.19$

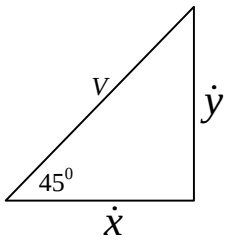
Question 6

(a) (i)

Intitial conditions:

$At\ t=0: \dot{x}=V\cos 45^0=\frac{V}{\sqrt{2}}$

$At\ t=0: \dot{y}=V\sin 45^0=\frac{V}{\sqrt{2}}$



In the horizontal plane

$\ddot{x}=0$

$\dot{x}=\frac{V}{\sqrt{2}}\ or\ \frac{\sqrt{2}}{2}V$

$x=\frac{\sqrt{2}}{2}Vt$

In the vertical plane:

$\ddot{y}=-g$

$\dot{y}=-gt+C$

$C=\frac{\sqrt{2}}{2}V$

$\therefore \dot{y}=-gt+\frac{\sqrt{2}}{2}V$

$y=-\frac{1}{2}gt^2+\frac{\sqrt{2}}{2}Vt+C_1$

$at\ t=0\ y=1$

$\therefore C_1=1$

$\therefore y=-\frac{1}{2}gt^2+\frac{\sqrt{2}}{2}Vt+1$

Now using

$x=\frac{\sqrt{2}}{2}Vt$

$2x=\sqrt{2}Vt \quad (note\ 2\div\sqrt{2}=\sqrt{2})$

$\sqrt{2}x=Vt$

$t=\frac{\sqrt{2}x}{V}$

sub this into y

$\therefore y=-\frac{1}{2}g\left(\frac{\sqrt{2}x}{V}\right)^2+\frac{\sqrt{2}}{2}V\left(\frac{\sqrt{2}x}{V}\right)+1$

$y=-\frac{1}{2}g\frac{2x^2}{V^2}+\frac{\sqrt{2}}{2}\frac{\sqrt{2}x}{1}+1$

$y=-\frac{gx^2}{V^2}+x+1$

$y=1+x-\frac{gx^2}{V^2}$

(a) (ii)	From the description we can see the ball passes through the point $(9.3, 2.3)$ $y = 1 + x - \frac{gx^2}{V^2}$ $\therefore 2.3 = 1 + 9.3 - \frac{9.8 \times 9.3^2}{V^2}$ $-8 = -\frac{847.602}{V^2}$ $V^2 = \frac{847.602}{8}$ $V \approx 10.8\ ms^{-2}$
(a) (iii)	y=0 when the ball hits the ground $\therefore 0 = 1 + x - \frac{9.8x^2}{10.3^2}$ $0 = 1 + x - \frac{9.8}{106}x^2$ $0 = 106 + 106x - 9.8x^2$ $\therefore x = \frac{-106 \pm \sqrt{106^2 - 4 \times 106 \times -9.8}}{2 \times 106}$ $x \approx 11.7$ (*disregarding* $-ve$ *value*) $\therefore$ distance from the net: $11.7 - 9.3 = 2.4m$
(b) (i)	$a = 3(1 - x^2)$ $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 3 - 3x^2$ $\therefore \frac{1}{2}v^2 = \int 3 - 3x^2 dx$ $\frac{1}{2}v^2 = 3x - x^3 + C$ At $x = 0, v = 4$ $\frac{1}{2}4^2 = 0 - 0^3 + C$ $C = 8$ $\frac{1}{2}v^2 = 3x - x^3 + 8$ $\therefore v^2 = 6x - 2x^3 + 16$ $v^2 = 16 + 6x - 2x^3$
(b) (ii)	Yes, the particle will return to the origin. Initially it is at the origin and moves to the right ($v=+4ms^{-1}$). When it goes past $x=1$ its acceleration becomes negative and thus the force will be operating back towards the origin. Its velocity on the second time through the origin will be $v=-4\ ms^{-1}$

Question 7

(a) (i)

$=(1+x)^{2n}+(1-x)^{2n}$

$={}^{2n}C_0+{}^{2n}C_1x^1+{}^{2n}C_2x^2+{}^{2n}C_3x^2+\ldots+{}^{2n}C_{2n}x^{2n}$

$+{}^{2n}C_0-{}^{2n}C_1x^1+{}^{2n}C_2x^2-{}^{2n}C_3x^2+\ldots+{}^{2n}C_{2n}x^{2n}$

$=2({}^{2n}C_0+{}^{2n}C_2x^2+{}^{2n}C_4x^4+\ldots+{}^{2n}C_{2n}x^{2n})$

(a) (ii)

For n=10 and x=1 the expression:

$(1+x)^{2n}+(1-x)^{2n}=2({}^{2n}C_0+{}^{2n}C_2x^2+{}^{2n}C_4x^4+\ldots+{}^{2n}C_{2n}x^{2n})$

$(1+1)^{20}+(1-1)^{20}=2({}^{20}C_0+{}^{20}C_2 1^2+{}^{20}C_4 1^4+\ldots+{}^{20}C_{20}1^{20})$

$2^{20}=2(1+{}^{20}C_2+{}^{20}C_4+\ldots+{}^{20}C_{20})$

$\therefore 2^{19}=1+{}^{20}C_2+{}^{20}C_4+\ldots+{}^{20}C_{20}$

(b) (i)

Let $y=\sqrt{x}$

$\therefore x=y^2$

$\therefore \frac{dx}{dy}=2y$

$\therefore dx=2ydy$

Now $\int \frac{dx}{\sqrt{x(1-x)}}=\int \frac{2ydy}{\sqrt{y^2(1-y^2)}}$

$=2\int \frac{dy}{\sqrt{(1-y^2)}}$

$=2\sin^{-1}y+C_1$

$=2\sin^{-1}\sqrt{x}+C_1$

(b) (ii)	Let $z = x - \frac{1}{2} \therefore x = z + \frac{1}{2}$ $\frac{dx}{dz} = 1 \therefore dx = dz$ Now $\int \frac{\mathrm{dx}}{\sqrt{\mathrm{x(1-x)}}} = \int \frac{dz}{\sqrt{\left(z+\frac{1}{2}\right)\left(1-z-\frac{1}{2}\right)}}$ $= \int \frac{dz}{\sqrt{\left(z+\frac{1}{2}\right)\left(\frac{1}{2}-z\right)}}$ $= \int \frac{2dz}{\sqrt{(2z+1)(1-2z)}}$ $= \int \frac{2dz}{\sqrt{1-4z^2}}$ $= \int \frac{2dz}{\sqrt{1-(2z)^2}}$ Let $u = 2z \therefore du = 2dz$ $\therefore$ $= \int \frac{du}{\sqrt{1-u^2}}$ $= \sin^{-1} u + C_2$ $= \sin^{-1} 2z + C_2$ $= \sin^{-1} 2(x - \frac{1}{2}) + C_2$ $= \sin^{-1}(2x-1) + C_2$
(b) (iii)	From (i) and (ii) $\sin^{-1}(2x-1) = 2\sin^{-1}\sqrt{x} + C$ To find C let x= $\frac{1}{2}$ $\sin^{-1} 0 = 2\sin^{-1}\sqrt{\frac{1}{2}} + C$ $0 = 2 \times \frac{\pi}{4} + C$ $\therefore C = -\frac{\pi}{2}$ $\therefore \sin^{-1}(2x-1) = 2\sin^{-1}\sqrt{x} - \frac{\pi}{2}$

(c)

$|4x-1| > 2\sqrt{x(1-x)}$

$(4x-1)^2 > 4x(1-x)$

$16x^2 - 8x + 1 > 4x - 4x^2$

$20x^2 - 12x + 1 > 0$

$(10x-1)(2x-1) > 0$

$\therefore x < \frac{1}{10}$ *or* $x > \frac{1}{2}$

Now $x(1-x) \geq 0$

$\therefore 0 \leq x \leq 1$

$\therefore 0 \leq x < \frac{1}{10}$ & $\frac{1}{2} < x \leq 1$

