1997 Higher School Certificate
Solutions
3 UNIT (ADDITIONAL) AND 3/4 UNIT (COMMON) MATHEMATICS

QUESTION 1

(a) $\dfrac{d}{dx}\left(e^{3x}\cos x\right) = 3e^{3x}\cos x + e^{3x}.(-\sin x)$

$= e^{3x}(3\cos x - \sin x).$

(b) The perpendicular distance from $(1, 2)$ to $3x - y + 4 = 0$ is

$$\left|\frac{3\times 1 - 1\times 2 + 4}{\sqrt{3^2+(-1)^2}}\right| = \frac{5}{\sqrt{10}}$$

$$= \frac{\sqrt{10}}{2} \text{ units.}$$

(c) $\log_a\left(\dfrac{b}{c}\right) = \log_a b - \log_a c$

$= 2{\cdot}8 - 4{\cdot}1$

$= -1{\cdot}3.$

(d) $\displaystyle\int_0^2 \frac{dx}{4+x^2} = \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2$

$= \frac{1}{2}\tan^{-1}\frac{2}{2} - \frac{1}{2}\tan^{-1}\frac{0}{2}$

$= \frac{1}{2}\times\frac{\pi}{4} - \frac{1}{2}\times 0$

$= \frac{\pi}{8}.$

(e) $u = 2x + 1$

$du = 2dx.$

$x = 1,\ u = 3$

$x = 0,\ u = 1$

$\therefore \displaystyle\int_0^1 \frac{4x}{2x+1}\,dx = \int_1^3 \frac{u-1}{u}\,du$

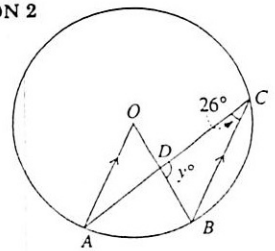
$= \displaystyle\int_1^3 1 - \frac{1}{u}\,du$

$= \left[u - \log_e u\right]_1^3$

$= (3 - \log_e 3) - (1 - \log_e 1)$

$= 2 - \log_e 3.$

QUESTION 2



(a) (i) $\angle AOB = 2\times\angle ACB$ (Angle at the centre is twice the angle at the circumference standing on the same arc.)

$= 2\times 26°$

$= 52°.$

(ii) $\angle CBO = \angle AOB$ (alternate $\angle$s; $AD \parallel BC$)

$\therefore \angle CBO = 52°$

$\angle BCD = 26°$ (given)

$\therefore \angle CDB = 102°$ (angle sum of ΔCDB)

$\therefore y = 102.$

(b) (i) $x = 3\sin(2t + 5).$

This is in the form $x = a\sin(nt + \alpha)$, where $a = 3$ and $n = 2$.

$$\text{Period} = \frac{2\pi}{n}$$

$$= \frac{2\pi}{2}$$

$$= \pi.$$

(ii) Since the motion is SHM,

$\ddot{x} = -n^2 x$

$\therefore \ddot{x} = -4x.$

Maximum acceleration is at $x = -3$.

$\therefore \ddot{x}_{max} = 12.$

OR

$\ddot{x} = -12\sin(2t + 5)$ and $-1 \le \sin\theta \le 1$,

so $\ddot{x}_{max} = 12.$

(iii) $v^2 = n^2\left(a^2 - x^2\right).$

$n = 2,\ a = 3,\ x = 2.$

$\therefore v^2 = 2^2\left(3^2 - 2^2\right)$

$\therefore v^2 = 20$

$\therefore v = \pm\sqrt{20}.$

$\therefore$ Speed is $\sqrt{20}$ or $2\sqrt{5}$.

OR

$\dot{x} = 6\cos(2t + 5).$

At $x = 2$, $2 = 3\sin(2t + 5)$

$\therefore \sin(2t + 5) = \frac{2}{3}$

$\therefore \cos(2t + 5) = \pm\sqrt{1 - \frac{4}{9}}$

$\therefore = \pm\frac{\sqrt{5}}{3}$

$\therefore \dot{x} = \pm 6\times\frac{\sqrt{5}}{3}$

$= \pm 2\sqrt{5}.$

$\therefore$ Speed is $2\sqrt{5}$.

(c) (i) $P(x) = x^3 + bx^2 + cx + d$

$P(0) = 0,\ \therefore d = 0$

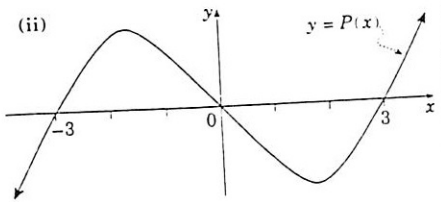
$P(3) = 0,\ \therefore 27 + 9b + 3c = 0$

$P(-3) = 0,\ \therefore -27 + 9b - 3c = 0.$

By inspection, $b = 0$ and $c = -9$.

$\therefore b = 0,\ c = -9$ and $d = 0$.

(ii)



(iii) $\dfrac{x^2 - 9}{x} > 0$ has the same solutions as $x(x^2 - 9) > 0$.

From the graph, $x(x^2 - 9) > 0$ if $-3 < x < 0$ or $x > 3$.

$\therefore$ Solution is $-3 < x < 0$ or $x > 3$.

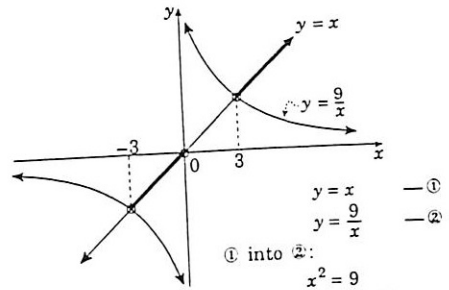
OR

$$\frac{x^2 - 9}{x} > 0$$

$$x - \frac{9}{x} > 0$$

$$x > \frac{9}{x}.$$

Draw $y = x$ and $y = \dfrac{9}{x}$ and find the x value where $y = x$ is above $y = \dfrac{9}{x}$.



$y = x$ —①

$y = \dfrac{9}{x}$ —②

① into ②:

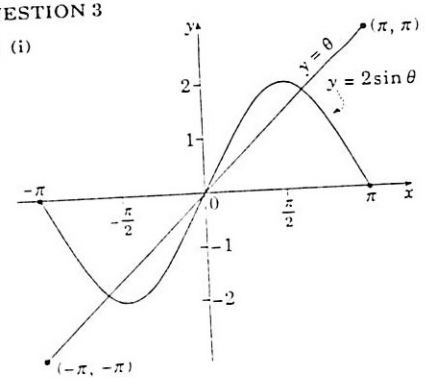
$x^2 = 9$

$\therefore x = 3, -3.$

Required solution is $-3 < x < 0$ or $x > 3$.

QUESTION 3

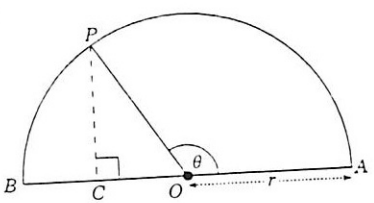
(a) (i)



(ii) The graphs of $y = 2\sin\theta$ and $y = \theta$ for $-\pi \le \theta \le \pi$ intersect three times.

$\therefore$ There are 3 solutions for $2\sin\theta = \theta$.

(b) (i)



Arc $PA = r\theta$.

$\therefore PC = OP\sin(\pi - \theta)$ (in ΔPCO)

$= r\sin\theta.$

As $PC = \frac{1}{2}.PA$

$\therefore r\sin\theta = \frac{1}{2}.r\theta$

$\therefore 2\sin\theta = \theta.$

(b) (ii) For $f(\theta) = 2\sin\theta - \theta$, a solution θ_2 is better than θ_1, where $\theta_2 = \theta_1 - \dfrac{f(\theta_1)}{f'(\theta_1)}$.

For $\theta_1 = 1{\cdot}8$:

$$\theta_2 = 1{\cdot}8 - \left(\frac{2\sin 1{\cdot}8 - 1{\cdot}8}{2\cos 1{\cdot}8 - 1}\right)$$
$$= 1{\cdot}8 - \frac{0{\cdot}147\,695}{-1{\cdot}454\,404}$$
$$= 1{\cdot}90, \text{ correct to 2 decimal places.}$$

(c) (i) $P(222) = P(2)\times P(2)\times P(2)$
$$= \frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}$$
$$= \frac{1}{216}.$$

(ii) P(exactly 2 twos)
= P(22− or 2−2 or −22), where − could be 1, 3, 4, 5, 6,
$$= 3\times\left\{\frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}\right\}$$
$$= \frac{5}{72}.$$

(iii) P(exactly 2 show the same number)
= P(exactly two 1's or 2's or ... or 6's)
$$= 6\times\frac{5}{72}$$
$$= \frac{5}{12}.$$

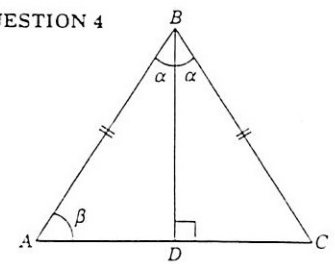
(iv) P(3 different numbers)
= 1 − {P(3 the same) + P(exactly 2 the same)}
$$= 1-\left\{\frac{6}{216}+\frac{5}{12}\right\}$$
$$= 1-\frac{96}{216}$$
$$= 1-\frac{4}{9}$$
$$= \frac{5}{9}.$$

For 9 games, we expect 5 to be 3 different numbers. The claim is correct.

OR

$$P(3 \text{ different}) = 1\times\frac{5}{6}\times\frac{4}{6} = \frac{5}{9}.$$

QUESTION 4



(a) (i) In ΔBAD, $\sin\beta = \dfrac{BD}{AB}$ and $\cos\alpha = \dfrac{BD}{AB}$.
$\therefore \sin\beta = \cos\alpha$.

(ii) In ΔABC, $\dfrac{\sin 2\alpha}{AC} = \dfrac{\sin\beta}{BC} = \dfrac{\sin\beta}{AB}$,
as $AB = BC$ (given).
$\therefore \sin 2\alpha = \dfrac{AC\sin\beta}{AB}$.
But $AC = 2AD$, as ΔABC is isosceles, and $BD \perp AC$,
$\therefore \sin 2\alpha = \dfrac{2AD\sin\beta}{AB}$.
Now, in ΔBAD, $\sin\alpha = \dfrac{AD}{AB}$,
$\therefore \sin 2\alpha = 2\sin\alpha\sin\beta$
$= 2\sin\alpha\cos\alpha$.

(iii) $\sin 2\alpha + \sin 2\alpha\cos^2\alpha + \sin 2\alpha\cos^4\alpha + \cdots$
GP: $a = \sin 2\alpha$, $r = \cos^2\alpha$.
$$S_\infty = \frac{a}{1-r}$$
$$= \frac{\sin 2\alpha}{1-\cos^2\alpha}$$
$$= \frac{2\sin\alpha\cos\alpha}{\sin^2\alpha}$$
$$= \frac{2\cos\alpha}{\sin\alpha}$$
$$= 2\cot\alpha.$$

(b) $x = \sin t$, $\therefore \dfrac{dx}{dt} = \cos t$.
$x = 0$, $t = 0$
$x = \frac{1}{2}$, $t = \frac{\pi}{6}$.
$$\therefore \int_0^{\frac{1}{2}}\sqrt{1-x^2}\,dx = \int_0^{\frac{\pi}{6}}\sqrt{1-\sin^2 t}\,.\cos t\,dt$$
$$= \int_0^{\frac{\pi}{6}}\cos^2 t\,dt$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{6}}(\cos 2t + 1)\,dt$$
$$= \frac{1}{2}\left[\frac{1}{2}\sin 2t + t\right]_0^{\frac{\pi}{6}}$$
$$= \frac{1}{2}\left(\frac{\sqrt{3}}{4}+\frac{\pi}{6}\right)-0$$
$$= \frac{\sqrt{3}}{8}+\frac{\pi}{12}.$$

OR

(ii) Aim to prove $\sum_{j=3}^{n}\binom{j-1}{2} = \binom{n}{3}$ for $n \geq 3$.

Step 1: If $n = 3$, LHS $= \binom{3-1}{2} = \binom{2}{2}$
$$= \frac{2!}{2!\,0!} = 1.$$
RHS $= \binom{3}{3}$
$$= \frac{3!}{3!\,0!} = 1.$$
$\therefore$ True for $n = 3$.

Step 2: Assume true for $n = k$, that is, assume $\sum_{j=3}^{k}\binom{j-1}{2} = \binom{k}{3}$ is true and aim to prove that $\sum_{j=3}^{k+1}\binom{j-1}{2} = \binom{k+1}{3}$ is true.

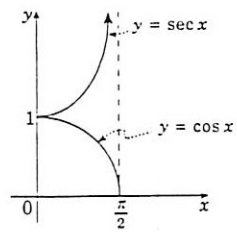
Then $\sum_{j=3}^{k+1}\binom{j-1}{2}$
$$= \sum_{j=3}^{k}\binom{j-1}{2} + \binom{k+1-1}{2}$$
$$= \binom{k}{3} + \binom{k}{2} \text{ by assumption}$$
$$= \binom{k+1}{3} \text{ from part (i).}$$

Hence true for $n = k+1$ if true for $n = k$.

Step 3: Since true for $n = k+1$ if true for $n = k$, and true for $n = 3$, then the statement is true for $n = 3+1 = 4$; $n = 4+1 = 5$; and so on.
By the principle of mathematical induction, it is true for all integers $n \geq 3$.

QUESTION 6

(a) (i) $f(x) = \sec x$ for $0 \leq x < \frac{\pi}{2}$.



Domain is $0 \leq x < \frac{\pi}{2}$.
Range is $y \geq 1$.
$\therefore$ Domain of inverse is $x \geq 1$.

(ii) Let $y = \sec x$.
Inverse is $x = \sec y$.
$$\frac{1}{x} = \cos y$$
$$\therefore \quad y = \cos^{-1}\left(\frac{1}{x}\right)$$
i.e. $f^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$.

(iii) $$\frac{d}{dx}f^{-1}(x) = \frac{-1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\cdot\left(-x^{-2}\right)$$
$$= \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}}$$
$$= \frac{1}{x\sqrt{x^2-1}}.$$

(b) (i) After 1st month, amount owing, in dollars:
$$B_1 = A + A\times\frac{r}{1200} - M$$
$$= A\left(1+\frac{r}{1200}\right) - M$$
$$= AR - M.$$

After 2nd month, amount owing, in dollars:
$$B_2 = (AR - M)R - M$$
$$= AR^2 - M(1+R).$$
$$\vdots$$

After nth month, amount owing, in dollars:
$$B_n = AR^n - M\left(1+R+R^2+\cdots+R^{n-1}\right)$$
$$= AR^n - M\times 1\times\left(\frac{R^n-1}{R-1}\right)$$
(sum of GP, where $a = 1$, $n = n$, $r = R$)
$$= AR^n - M\left(\frac{R^n-1}{R-1}\right).$$

(ii) $n = 60$, so $B_{60} = 0$.
$$\therefore M\left(\frac{R^{60}-1}{R-1}\right) = 30\,000\times R^{60},$$
where $R = 1+\frac{9}{1200} = 1\frac{3}{400} = 1{\cdot}0075$
$$\therefore \quad M = \frac{30\,000\times 1{\cdot}0075^{60}\times 0{\cdot}0075}{1{\cdot}0075^{60}-1}$$
$$= 622{\cdot}75.$$

The monthly repayments should be \$622·75.

(iii) After 12th normal payment, amount owing, in \$, is:

$$B_{12} = 30\,000 \times 1{\cdot}0075^{12} - 622{\cdot}75 \times \left(\frac{1{\cdot}0075^{12} - 1}{1{\cdot}0075 - 1}\right),$$

$$= \$25\,025{\cdot}11.$$

Now another \$5000 is paid, so \$20 025·11 is owing.

This means that after a *further* k repayments, amount owing, in \$, is:

$$B_k = 20\,025{\cdot}11 \times 1{\cdot}0075^k - 622{\cdot}75\,\frac{1{\cdot}0075^k - 1}{0{\cdot}0075}.$$

We want k, so that $B_k = 0$.

$$\therefore\ 0{\cdot}0075 \times 20\,025{\cdot}11 \times 1{\cdot}0075^k = 622{\cdot}75\left(1{\cdot}0075^k - 1\right)$$

$$150{\cdot}188\,325 \times 1{\cdot}0075^k = 622{\cdot}75 \times 1{\cdot}0075^k - 622{\cdot}75.$$

$$\therefore\ 622{\cdot}75 = 472{\cdot}561\,675 \times 1{\cdot}0075^k$$

$$1{\cdot}0075^k = 1{\cdot}317\,817\,404$$

$$\therefore\ k = \frac{\ln 1{\cdot}317\,817\,404}{\ln 1{\cdot}0075}$$

$$= 36{\cdot}935.$$

∴ Another 37 repayments will be needed.

QUESTION 7

(a) (i) Initially, when $t = 0$, $x = 0$, $y = h$, $\dot{x} = V$, $\dot{y} = 0$ as particle is projected horizontally.

$\ddot{x} = 0$

Integrating w.r.t. t, $\dot{x} = C_1$.

When $t = 0$, $\dot{x} = V$,

$\therefore\ \dot{x} = V$.

Integrating w.r.t. t, $x = Vt + C_2$.

When $t = 0$, $x = 0$,

$\therefore\ x = Vt$.

$\ddot{y} = -g$.

Integrating w.r.t. t, $\dot{y} = -gt + C_3$.

When $t = 0$, $\dot{y} = 0$,

$\therefore\ y = -gt$.

Integrating w.r.t. t, $y = -\frac{1}{2}gt^2 + C_4$.

When $t = 0$, $y = h$,

$\therefore\ y = -\frac{1}{2}gt^2 + h$

$= h - \frac{1}{2}gt^2$.

(ii) $V = 216$ km/h $= \frac{216\,000}{3600}$ m/s $= 60$ m/s.

$h = 120$ m, $g = 10$ m/s^2.

Canister hits the water when $y = 0$.

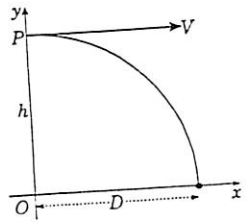
$\therefore\ 0 = 120 - 5t^2$

$\therefore\ t^2 = 24$

$\therefore\ t = \sqrt{24} = 2\sqrt{6}$ s $\quad (\because t \neq -2\sqrt{6})$.

∴ It hits the water after $2\sqrt{6}$ seconds.

(iii)



Sailor drifts at speed of 3·6 km/h

$= \frac{3{\cdot}6 \times 1000}{3600}$ m/s

$= 1$ m/s.

∴ Sailor drifts a distance of $2\sqrt{6}$ metres ($\approx 4{\cdot}9$ m) in the time it takes for the canister to hit the water.

Now, when $t = 2\sqrt{6}$, $x = Vt = 60 \times 2\sqrt{6}$

$\therefore\ x = 120\sqrt{6}$ m. $(\approx 293{\cdot}94$ m)

Without drifting,

$120\sqrt{6} - 50 \leq D \leq 120\sqrt{6} + 50$.

∴ *With* drifting,

$118\sqrt{6} - 50 \leq D \leq 118\sqrt{6} - 50$.

That is, D can take approx. values between 239 m and 339 m.

(b) (i) Consider expansion of $(1+x)^{n-1}$:

$$(1+x)^{n-1} = 1 + \binom{n-1}{1}x + \binom{n-1}{2}x^2 + \cdots + \binom{n-1}{n-2}x^{n-2} + x^{n-1}.$$

Put $x = 1$,

$$\therefore\ 2^{n-1} = 1 + \binom{n-1}{1} + \binom{n-1}{2} + \cdots + \binom{n-1}{n-2} + 1.$$

$$\therefore\ \binom{n-1}{1} + \binom{n-1}{2} + \cdots + \binom{n-1}{n-2} = 2^{n-1} - 2.$$

$$\therefore\ n\left[\binom{n-1}{1} - \binom{n-1}{2} + \cdots + \binom{n-1}{n-2}\right] = n\left(2^{n-1} - 2\right).$$

(ii) We require n such that

$n\left(2^{n-1} - 2\right) > 20\,000$.

$2n\left(2^{n-2} - 1\right) > 20\,000$

$2^{n-2} - 1 > \frac{10\,000}{n}$ (n +ve integer).

We could do it by trial and error:

$n = 10$, LHS = 255, RHS = 1000, $\therefore\ n \neq 10$.

$n = 11$, LHS = 511, RHS = 909·09̇, $\therefore\ n \neq 11$.

$n = 12$, LHS = 1023, RHS = 833·3̇, $\therefore\ n = 12$.

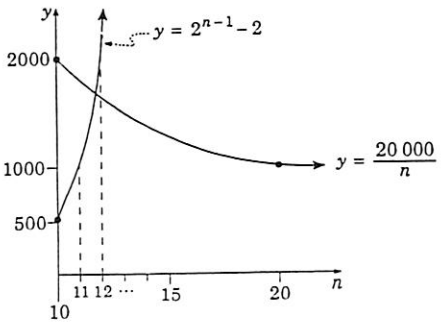
OR

We could do it graphically:

If $n\left(2^{n-1} - 2\right) > 20\,000$,

then $2^{n-1} - 2 > \frac{20\,000}{n}$ (as n is +ve).

$\left[\text{Consider } n = 10;\ 2^9 - 2 > \frac{20\,000}{10}.\right]$



From the graph, $n = 12$ is the first value.

END OF 3 UNIT (ADDITIONAL) AND 3/4 UNIT (COMMON) MATHEMATICS SOLUTIONS