1997 HSC Full Solutions

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QUESTION 1

(a)

$$\frac{d}{dx}(e^{3x}\cos x) = e^{3x}(-\sin x) + \cos x.3e^{3x}$$
$$= e^{3x}(3\cos x - \sin x)$$

(b)

Point (1,2) and the line 3x-y+4=0

$$d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$
$$= \left|\frac{3\times 1 - 1\times 2 + 4}{\sqrt{9+1}}\right|$$
$$= \left|\frac{5}{\sqrt{10}}\right| = \frac{5}{\sqrt{10}} = \frac{5\sqrt{10}}{10} = \frac{\sqrt{10}}{2}$$

(c)

$\log_a b = 2.8 \;\; \log_a c = 4.1$

$$\therefore \log_a \frac{b}{c} = \log_a b - \log_a c$$
$$= 2.8 - 4.1 = -1.3$$

(d)

$$\int_0^2 \frac{1}{4+x^2}dx = \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2$$
$$= \left(\frac{1}{2}\tan^{-1}\frac{2}{2}\right) - \left(\frac{1}{2}\tan^{-1}\frac{0}{2}\right)$$
$$= \frac{1}{2}\times\frac{\pi}{4} = \frac{\pi}{8}$$

(e)

$Let \;\; u = 2x+1$

$$\therefore du = 2dx$$

$$\& \; x = \frac{u-1}{2}$$

$at \; x = 0, u = 1$

$x = 1, u = 3$

$$\therefore \int_0^1 \frac{4x}{2x+1}dx = \int_0^1 \frac{2x\times 2dx}{2x+1}$$
$$= \int_1^3 \frac{u-1}{u}du$$
$$= \int_1^3 1 - \frac{1}{u}.du$$
$$= \left[u - \ln u\right]_1^3$$
$$= (3 - \ln 3) - (1 - \ln 1)$$
$$= 2 - \ln 3$$

Question 2

(a) (i)

The angle at the centre AOB is twice the angle on the circumference ACB

(a)(ii)

$\angle AOB=52^0$ (from part (i))
$\angle OAC=26^0$ (alt angles on || lines are =)
$y=52^0+26^0=78^0$ (ext $\angle$ of a triangle = sum of opp. Two int. $\angle$'s)

(b) (i)

$$Period = \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$$

(b) (ii)

$$\begin{aligned} \dot{x} &= 3\cos(2t+5)\times 2 \\ &= 6\cos(2t+5) \\ \ddot{x} &= -6\sin(2t+5)\times 2 \\ &= -12\sin(2t+5) \end{aligned}$$

$\therefore$ maximum acceleration occurs when $\sin(2t+5) = -1$

$\ddot{x}_{max} = 12$

(b) (iii)

$$x = 3\sin(2t+5)$$

Now $2 = 3\sin(2t+5)$

$$\therefore \sin(2t+5) = \frac{2}{3}$$

$$\therefore \cos(2t+5) = \frac{\sqrt{5}}{3}$$

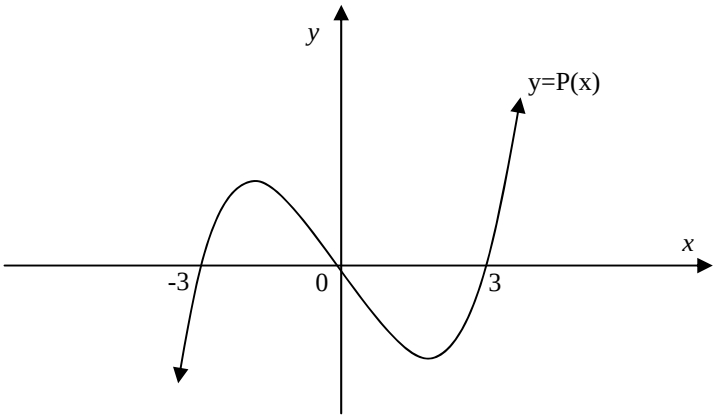
Since $\sin(2t+5) = \frac{2}{3} = \frac{\text{opposite}}{\text{hypotenuse}}$

then adjacent $= \pm\sqrt{3^2-2^2} = \pm\sqrt{5}$

Now $\dot{x} = 6\cos(2t+5)$

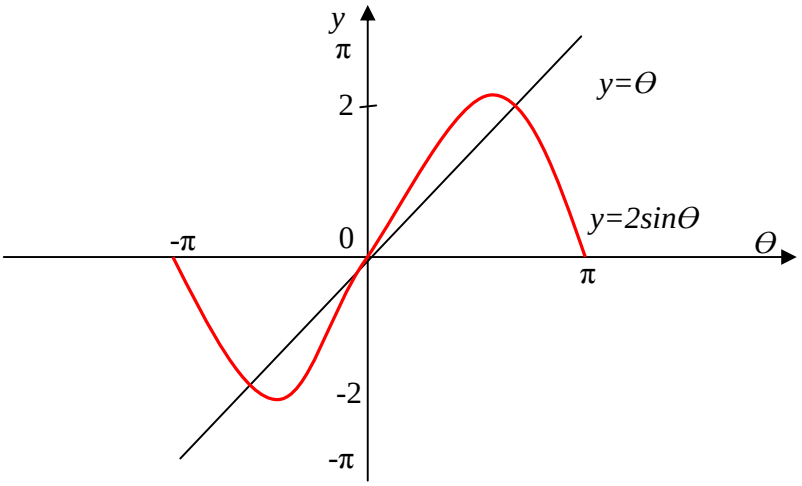
$$= 6\times\frac{\pm\sqrt{5}}{3} = \pm 2\sqrt{5}$$

$\therefore$ *speed* $= 2\sqrt{5}$

(c) (i)	$P(x) = x^3 + bx^2 + cx + d$ Since x=0 is a root then P(0)=0+0+0+d=0 $\therefore d = 0$ $\therefore P(x) = x^3 + bx^2 + cx$ $P(-3) = -3^3 + b(-3)^2 + c(-3)$ *i.e.* $0 = -27 + 9b - 3c$ ...(1) $P(3) = 3^3 + b(3)^2 + c(3)$ *i.e.* $0 = 27 + 9b + 3c$ ...(2) $(1) + (2)$ $0 = 18b$ $b = 0$ *sub* $b = 0$ into (2) 0=27+0+3c $c = -9$ $\therefore b = 0; c = -9; d = 0$
(c)(ii)	$P(x) = x^3 - 9x = x(x^2 - 9) = x(x-3)(x+3)$ 
(c)(iii)	$\frac{x^2 - 9}{x} > 0$ $\times x^2$ *both sides* $\frac{x^2 - 9}{x} \times x^2 > 0 \times x^2$ $x(x^2 - 9) > 0$ *from the graph* $-3 < x < 0$ *or* $x > 3$

Question 3

(a) (i)



(a) (ii) From the graph the intersection points are the solutions. Therefore three solutions exist.

(b) (i)

The arc length of AP=rθ

The arc length of AP=2PC

$\therefore$ 2PC=rθ

$$Now: \sin\theta = \frac{PC}{r}$$

$$\therefore r\sin\theta = PC$$

$$\therefore 2r\sin\theta = \mathrm{r}\theta$$

$$\therefore 2\sin\theta = \theta$$

(b) (ii)

$$Let: f(\theta) = 2\sin\theta - \theta$$

$$\theta_2 = \theta_1 - \frac{f(\theta)}{f'(\theta)}$$

$$= 1.8 - \frac{f(1.8)}{f'(1.8)}$$

$$= 1.8 - \frac{2\sin 1.8 - 1.8}{2\cos 1.8 - 1}$$

$$\approx 1.9$$

(c)(i)

$$\mathrm{P}(2, 2, 2) = \left(\frac{1}{6}\right)^3 = \frac{1}{216}$$

(c)(ii)

$$P(exactly\ 2\ dice\ show\ 6) = {}^3C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right) = \frac{5}{72}$$

(c)(iii)

$$\mathrm{P}(2\text{ dice show the same number}) = 6\ {}^3C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right) = \frac{5}{12}$$

(c)(iv)	$P(\textit{three different numbers}) = 1 - P(\textit{any triple}) - P(\textit{any double})$ $= 1 - 6 \times \frac{1}{216} - \frac{5}{12} = \frac{5}{9}$, which is more than $\frac{1}{2}$ The player was right. Alternatively, P(3 different numbers) = $\frac{6}{6} \times \frac{5}{6} \times \frac{4}{6} = \frac{5}{9}$

Question 4

(a) (i)

$$\sin\beta = \frac{BD}{AB}$$

$$\cos\alpha = \frac{BD}{AB}$$

$$\therefore \sin\beta = \cos\alpha$$

(a) (ii)

Using the Sine Rule

$$\frac{\sin 2\alpha}{AC} = \frac{\sin\beta}{BC}$$

$$\sin 2\alpha = \frac{AC\sin\beta}{BC} \quad \text{note AC=2DC}$$

$$= \frac{2DC\sin\beta}{BC} \quad \text{note } \sin\alpha = \frac{DC}{BC}$$

$$= 2\sin\alpha\sin\beta \quad \text{note } \sin\beta = \cos\alpha \quad \text{part (i)}$$

$$= 2\sin\alpha\cos\alpha$$

(a) (iii)

$$Sum = \sin 2\alpha + \sin 2\alpha\cos^2\alpha + \sin 2\alpha\cos^4\alpha + \sin 2\alpha\cos^6\alpha + \ldots$$

$$\therefore T_1 = a = \sin 2\alpha \text{ and } r = \cos^2\alpha$$

$$Since \ \ 0 < \cos^2\alpha < 1 \ then \ S_\infty = \frac{a}{1-r}$$

$$\therefore S_\infty = \frac{\sin 2\alpha}{1-\cos^2\alpha}$$

$$= \frac{\sin 2\alpha}{\sin^2\alpha}$$

$$= \frac{2\sin\alpha\cos\alpha}{\sin^2\alpha}$$

$$= \frac{2\cos\alpha}{\sin\alpha}$$

$$= 2\cot\alpha$$

(b)

$$\int_0^{\frac{1}{2}} \sqrt{1-x^2}\,dx$$

$Let\ x = \sin t$

$$\therefore x = 0;\ t = 0\ \&\ x = \frac{1}{2}; t = \frac{\pi}{6}$$

$$\frac{dx}{dt} = \cos t$$

$$dx = \cos t dt$$

$$\therefore \int_0^{\frac{1}{2}} \sqrt{1-x^2}\,dx = \int_0^{\frac{\pi}{6}} \sqrt{1-\sin^2 t}\cos t dt$$

$$= \int_0^{\frac{\pi}{6}} \sqrt{\cos^2 t}\cos t dt$$

$$= \int_0^{\frac{\pi}{6}} \cos^2 t dt$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{2}\left(1+\cos 2t\right) dt$$

$$= \frac{1}{2}\left[t + \frac{1}{2}\sin 2t \right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{2}\left(\frac{\pi}{6} + \frac{1}{2}\sin 2\frac{\pi}{6} \right) - \frac{1}{2}\left(0-0\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{6} + \frac{1}{2}\times\frac{\sqrt{3}}{2} \right)$$

$$= \frac{1}{4}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} \right)$$

(c)(i)

$$\tan\theta = \frac{0.9}{x}$$

$$\therefore x = \frac{0.9}{\tan\theta}$$

$$\therefore \frac{dx}{dt} = \frac{-0.9\sec^2\theta}{\tan^2\theta}$$

$$= \frac{-0.9\dfrac{1}{\cos^2\theta}}{\dfrac{\sin^2\theta}{\cos^2\theta}}$$

$$= \frac{-0.9}{\sin^2\theta}$$

(c)(ii)	$Now \frac{dx}{dt} = -240$ $\frac{dx}{d\theta} = \frac{-0.9}{\sin^2\theta}$ & $\frac{d\theta}{dt} = \frac{d\theta}{dx}.\frac{dx}{dt}$ $= \frac{\sin^2\theta}{-0.9}.-240$ $= \frac{240\sin^2\theta}{0.9}$ $at\ \theta = \frac{\pi}{4}$ $\frac{d\theta}{dt} = \frac{240\sin^2\frac{\pi}{4}}{0.9} = \frac{240.\frac{1}{2}}{0.9} = \frac{120}{0.9} = \frac{1200}{9} = \frac{400}{3}$ radians/hour $= \frac{400}{3\times60\times60}$ radians/second $= \frac{1}{27}$ radians/second

Question 5

(a)(i)

$v = \sqrt{2x}e^{-x^2}, \quad x \geq 0.1$

$a = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

$a = \dfrac{d}{dx}\left(\dfrac{1}{2}\left(\sqrt{2x}e^{-x^2}\right)^2\right)$

$a = \dfrac{d}{dx}\left(\dfrac{1}{2}2xe^{-2x^2}\right)$

$= \dfrac{d}{dx}\left(xe^{-2x^2}\right)$

$= x.-4xe^{-2x^2} + e^{-2x^2}$

$= e^{-2x^2}(1-4x^2)$

(a)(ii)

Fastest speed occurs when a=0

$a = e^{-2x^2}(1-4x^2)$

$0 = e^{-2x^2}(1-4x^2)$

$(1-4x^2) = 0$

$1 = 4x^2$

$x^2 = \dfrac{1}{4}$

$x = \dfrac{1}{2}$ since x ≥ 0.1

$v_{\max} = \sqrt{2\dfrac{1}{2}e^{-\frac{1}{2}^2}}$

$= e^{-\frac{1}{4}}$

(a)(iii)

$v = \sqrt{2x}e^{-x^2}$

$\dfrac{dx}{dt} = \sqrt{2x}e^{-x^2} = \dfrac{\sqrt{2}\sqrt{x}}{e^{x^2}}$

$\dfrac{dt}{dx} = \dfrac{e^{x^2}}{\sqrt{2}x^{\frac{1}{2}}}$

$\therefore t = \int \dfrac{e^{x^2}}{\sqrt{2}x^{\frac{1}{2}}}.dx$

$t = \int \dfrac{1}{\sqrt{2}x^{\frac{1}{2}}}.e^{x^2}dx$

Let T = the time from x=1 to x=2

$\therefore T = \int_1^2 \dfrac{1}{\sqrt{2}x^{\frac{1}{2}}}.e^{x^2}dx$

(a)(iv)	$T \quad = \int_1^2 \frac{1}{\sqrt{2}x^{\frac{1}{2}}}.e^{x^2}\,dx$ $T(x) = \frac{0.5}{2}(y_0 + 2y_1 + y_3)$ $= 0.25\left(\frac{1}{\sqrt{2}\times 1^{\frac{1}{2}}}.e^{1} + 2\left(\frac{1}{\sqrt{2}\times 1.5^{\frac{1}{2}}}.e^{1.5^2}\right) + \frac{1}{\sqrt{2}\times 2^{\frac{1}{2}}}.e^{4}\right)$ $= 0.25(1.9221 + 2\times 5.4778 + 27.2991)$ ≈ 10 $\therefore$ Time taken to travel from x=1 to x=2 is approximately 10 seconds
(b) (i)	$\binom{n}{r} + \binom{n}{r+1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)!(n-r-1)!}$ $= \frac{n!(r+1)}{(r+1)!(n-r)!} + \frac{n!(n-r)}{(r+1)!(n-r)!} = \frac{n!(r+1+n-r)}{(r+1)!(n-r)!}$ $= \frac{n!(n+1)}{(r+1)!(n-r)!} = \frac{(n+1)!}{(r+1)!(n-r)!} = \binom{n+1}{r+1}$

(b) (ii)

$$\sum_{j=3}^{n}\binom{j-1}{2}=\binom{n}{3}$$

Step 1 Prove true for n=3

$$LHS=\sum_{j=3}^{3}\binom{3-1}{2}=\binom{2}{2}=1$$

$$RHS=\binom{3}{3}=1$$

True for first value of n

Step 2: Assume true for say n=k

i.e. $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{k-1}{2}=\binom{k}{3}$

Show true for k+1

i.e. $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{k-1}{2}+\binom{k}{2}=\binom{k+1}{3}$

$$LHS=\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{k-1}{2}+\binom{k}{2}$$

$$=\binom{k}{3}+\binom{k}{2}$$

$$=\binom{k+1}{3}\ \textit{from part } (i)$$

$$=RHS$$

Step 3: "We know the formula is true for n = 3 (step 1), so it is true for n = 4 (step 2). Since it is true for n = 4, then it is true for n = 5, and so on.

Therefore it is true for any positive integer $n \geq 3$"

Question 6

(a)(i)

$f : y = \sec x$

$Domain : 0 \le x < \pi$

$Range : y \ge 1$

Doamin and range for a function and its inverse swap

$f^{-1} : x = \sec y$

$Range : 0 \le y < \pi$

$Domain : x \ge 1$

(a) (ii)

$f : y = \sec x$

$f^{-1} : x = \sec y$

$\therefore x = \dfrac{1}{\cos y}$

$\therefore \cos y = \dfrac{1}{x}$

$\therefore f^{-1} : y = \cos^{-1}\left(\dfrac{1}{x}\right)$

(a) (iii)

$f^{-1} : y = \cos^{-1}\left(\dfrac{1}{x}\right)$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\frac{1}{x^2}}} \times -\frac{1}{x^2}$$

$$= \frac{1}{x^2\sqrt{\frac{x^2-1}{x^2}}}$$

$$= \frac{1}{x\sqrt{x^2-1}} \quad (\text{note } x \neq 1)$$

(b) (i)

(b) (ii)

(b) (iii)

Question 7

(a) (i)

Intitial conditions:

$At\ t=0: \dot{x}=V, x=0$

$At\ t=0: \dot{y}=0, y=h$

In the horizontal plane

$\ddot{x}=0$

$\dot{x}=V$

$x=Vt$

In the vertical plane:

$\ddot{y}=-g$

$\dot{y}=-gt+C$

$C=0$

$\therefore \dot{y}=-gt$

$y=-\frac{1}{2}gt^2+C_1$

$\therefore C_1=h$

$\therefore y=-\frac{1}{2}gt^2+h$

$y=h-\frac{1}{2}gt^2$

(a) (ii)

$V=216km/h\ =\frac{216\times1000}{60\times60}=60ms^{-1}$

The cannister hits the ground when y=0

We are given h=120 and g=-10

$\therefore 0=120-\frac{1}{2}.10.t^2$

$120=5t^2$

$t^2=24$

$t=\sqrt{24}\approx4.9s$

(a) (iii)

The cannister travels in 4.9s in the horizontal direction

x=Vt

$x=60\times4.9=294m$

The sailor is moving at $3.6km/h=\frac{3.6\times1000}{60\times60}=1m/s$

$\therefore$ In 4.9s the sailor drifts 4.9m

$\therefore$ The sailors distance from O $=\ D+4.9m$

$\therefore$ To land at most 50m from the sailor

$294-50\le D+4.9\le294+50$

$244\le D+4.9\le344$

$239.1\le D\le339.1$

(b) (i)	$n\binom{n-1}{1}+n\binom{n-1}{2}+\ldots+n\binom{n-1}{n-2}=n\left(\binom{n-1}{1}+\binom{n-1}{2}+\ldots+\binom{n-1}{n-2}\right)$ $(1+x)^{n-1}=\binom{n-1}{0}+\binom{n-1}{1}x+\binom{n-1}{2}x^2+\ldots+\binom{n-1}{n-2}x^{n-2}+\binom{n-1}{n-1}x^{n-1}$ $=1+\binom{n-1}{1}x+\binom{n-1}{2}x^2+\ldots+\binom{n-1}{n-2}x^{n-2}+x^{n-1}$ Put $x=1$: $2^{n-1}=1+\binom{n-1}{1}+\binom{n-1}{2}+\ldots+\binom{n-1}{n-2}+1$ $\therefore\binom{n-1}{1}+\binom{n-1}{2}+\ldots+\binom{n-1}{n-2}=2^{n-1}-2$ $\therefore n\left(\binom{n-1}{1}+\binom{n-1}{2}+\ldots+\binom{n-1}{n-2}\right)=n\left(2^{n-1}-2\right)$
(b) (ii)	Solving $n\left(2^{n-1}-2\right)>20000$ $2n\left(2^n-1\right)>20000$ $n\left(2^n-1\right)>10000$ When $n=11, n\left(2^n-1\right)=5621$ When $n=12, n\left(2^n-1\right)=12276$ $\therefore n=12$