1996 HIGHER SCHOOL CERTIFICATE
SOLUTIONS
3 UNIT (ADDITIONAL) AND 3/4 UNIT (COMMON) MATHEMATICS

QUESTION 1

(a) $P(x) = 2x^3 + x + a$

$(x-2)$ is a factor, $\therefore \quad P(2) = 0$

$P(2) = 2(2^3) + 2 + a$

$= 18 + a$

$= 0$ if $a = -18$.

(b) m_1: gradient of $2x + y = 4$ is -2

m_2: gradient of $x - y = 2$ is 1

If θ is the acute angle between the lines,

then $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

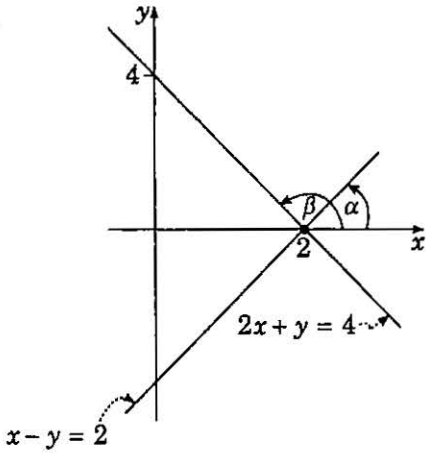
$= \left|\dfrac{-2-1}{1+(-2)(+1)}\right|$

$= \left|\dfrac{-3}{-1}\right|$

$= 3$

$\therefore \quad \theta = 72°$ to the nearest degree.

OR



$2x + y = 4$: gradient $= -2$

$\therefore \quad \beta = 116{\cdot}565°$

$x - y = 2$: gradient $= 1$

$\therefore \quad \alpha = 45°$

$\therefore$ acute $\angle$ between $= 116{\cdot}565 - 45$

$= 72°$ (to nearest degree).

(c) Ratio $3 : -1$ since division is external

$A(-1, 2)$, $B(3, 5)$.

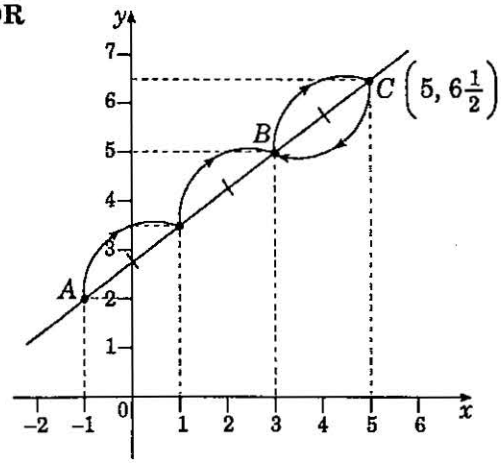
$k : \ell = 3 : -1$

$$C = \left(\frac{kx_2 + \ell x_1}{k+\ell}, \frac{ky_2 + \ell y_1}{k+\ell}\right)$$

$$= \left(\frac{3\times 3 - 1\times -1}{3-1}, \frac{3\times 5 - 1\times 2}{3-1}\right)$$

$$= \left(5, 6\tfrac{1}{2}\right).$$

OR



(d) Choose 3 from 8 and 4 from 6.

$\therefore$ No. ways $= {}^8C_3 \times {}^6C_4$.

(e) $\dfrac{2}{x-1} \le 1$

$x \ne 1$

When $x - 1 > 0$, i.e. $x > 1$

$2 \le x - 1$

$x \ge 3$.

} that is, $x \ge 3$

When $x - 1 < 0$, i.e. $x < 1$

$2 \ge x - 1$

$x \le 3$.

} that is, $x < 1$

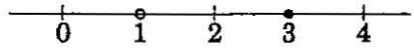
$\therefore \quad x \ge 3$ or $x < 1$.

OR Using the critical points method:

Consider $\dfrac{2}{x-1} = 1$ (noting $x \neq 1$)

$2 = x-1$

$x = 3$

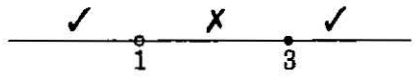


Now consider $\dfrac{2}{x-1} \leq 1$

Test $x = 0$ $\quad \dfrac{2}{-1} \leq 1$ TRUE

$x = 2$ $\quad \dfrac{2}{1} \leq 1$ FALSE

$x = 4$ $\quad \dfrac{2}{4} \leq 1$ TRUE



Solution is $x < 1$ or $x \geq 3$.

(f)

$u = e^x$

$\dfrac{du}{dx} = e^x$

$du = e^x\,dx$

$$\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{du}{1+u^2}$$

$= \tan^{-1} u + c$

$= \tan^{-1} e^x + c.$

QUESTION 2

(a) $f(x) = x^3 - \ln(x+1)$

(i) $f(0{\cdot}8) = 0{\cdot}8^3 - \ln(0{\cdot}8+1)$

$\approx -0{\cdot}076$

$f(0{\cdot}9) = 0{\cdot}9^3 - \ln(0{\cdot}9+1)$

$\approx 0{\cdot}087$

Since $f(0{\cdot}8)$ and $f(0{\cdot}9)$ are opposite in sign and $y = f(x)$ is a continuous curve between $x = 0{\cdot}8$ and $x = 0{\cdot}9$, then there is a root between $x = 0{\cdot}8$ and $x = 0{\cdot}9$.

(ii) Let $x_1 = \dfrac{0{\cdot}8+0{\cdot}9}{2}$

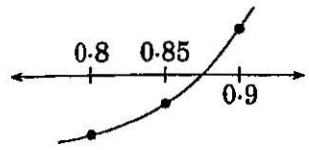
$= 0{\cdot}85$

$f(x_1) = 0{\cdot}85^3 - \ln(0{\cdot}85+1)$

$\approx -0{\cdot}00106\ldots$

$< 0.$

$\therefore$ Root is between 0·85 and 0·9.



That is, root is closer to 0·9 than 0·8, so its value to one decimal place is 0·9.

(b) Using $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$.

$$\int_6^{15} \frac{dx}{\sqrt{x^2+64}}$$

$= \left[\ln\left(x+\sqrt{x^2+64}\right)\right]_6^{15}$

$= \ln\left(15+\sqrt{15^2+64}\right) - \ln\left(6+\sqrt{6^2+64}\right)$

$= \ln 32 - \ln 16$

$= \ln\dfrac{32}{16}$

$= \ln 2.$

(c) **(i)** $OS = OP$ (radius of circle centre O)

$OP = PS$ (radius of circle centre P)

$\therefore$ $OS = SP = OP$

$\therefore$ ΔSOP is equilateral.

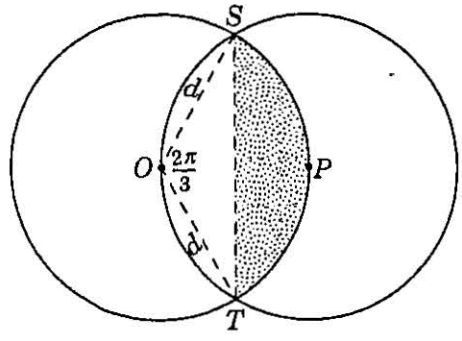
(ii) Similarly ΔTOP is equilateral

$\therefore$ $\angle SOT = \angle SOP + \angle POT$

$= \dfrac{\pi}{3} + \dfrac{\pi}{3}$

$= \dfrac{2\pi}{3}.$

(iii)



Area of region

$= 2$(area segment SPT)

$= 2\left(\text{(area sector } OST) - \text{(area } \Delta OST)\right).$

Area sector $OST = \frac{1}{2}d^2.\frac{2\pi}{3}$.

Area triangle $OST = \frac{1}{2}d^2.\sin\frac{2\pi}{3}$

$= \frac{1}{2}d^2.\frac{\sqrt{3}}{2}$.

$\therefore$ Area required $= 2\left(\frac{d^2\pi}{3} - \frac{d^2\sqrt{3}}{4}\right)$

$= \frac{2d^2\pi}{3} - \frac{\sqrt{3}d^2}{2}$

$= d^2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$.

QUESTION 3

(a) **(i)** Function is even if $f(-x) = f(x)$.

Here $f(x) = \frac{1}{\sqrt{16-x^2}}$

$f(-x) = \frac{1}{\sqrt{16-(-x)^2}}$

$= \frac{1}{\sqrt{16-x^2}}$

$= f(x)$

$\therefore$ function is even.

(ii) Area $= \int_{-2}^{2} \frac{1}{\sqrt{16-x^2}}\,dx$

$= 2\int_{0}^{2} \frac{1}{\sqrt{16-x^2}}\,dx$

$= 2\left[\sin^{-1}\frac{x}{4}\right]_0^2$

$= 2\left[\sin^{-1}\frac{2}{4} - \sin^{-1}\frac{0}{4}\right]$

$= 2\left[\sin^{-1}\frac{1}{2} - \sin^{-1}0\right]$

$= 2\left[\frac{\pi}{6} - 0\right]$

$= \frac{\pi}{3}$ units^2.

(b) $\cos 2\theta = 2\cos^2\theta - 1$

$\therefore$ $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$

$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 x\,dx = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos 2x + 1)\,dx$

$= \frac{1}{2}\left[\frac{1}{2}\sin 2x + x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

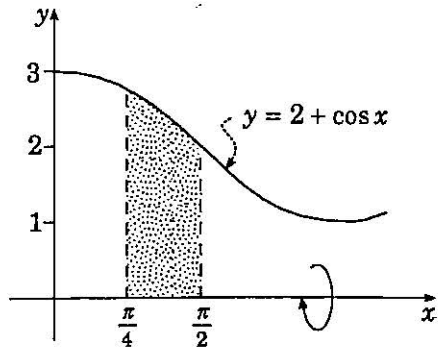
$= \frac{1}{2}\left[\left(\frac{1}{2}\sin\left(2\times\frac{\pi}{2}\right) + \frac{\pi}{2}\right) - \left(\frac{1}{2}\sin\left(2\times\frac{\pi}{4}\right) + \frac{\pi}{4}\right)\right]$

$= \frac{1}{2}\left[\frac{1}{2}\times 0 + \frac{\pi}{2} - \frac{1}{2}\times 1 - \frac{\pi}{4}\right]$

$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{2}\right]$

$= \left(\frac{\pi}{8} - \frac{1}{4}\right)$ units^2.

(c)



Vol. $= \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2+\cos x)^2\,dx$ using $V = \pi\int_a^b y^2\,dx$

$= \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(4 + 4\cos x + \cos^2 x\right)dx$

$= \pi\left[4x + 4\sin x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 x\,dx$

$= \pi\left[\left(\frac{4\pi}{2} + 4\sin\frac{\pi}{2}\right) - \left(\frac{4\pi}{4} + 4\sin\frac{\pi}{4}\right)\right]$

$+ \pi\left[\frac{\pi}{8} - \frac{1}{4}\right]$ (using the result from (b))

$= \pi\left[(2\pi + 4) - \left(\pi + \frac{4}{\sqrt{2}}\right) + \left(\frac{\pi}{8} - \frac{1}{4}\right)\right]$

$= \pi\left[2\pi - \pi + \frac{\pi}{8} + 4 - \frac{1}{4} - 2\sqrt{2}\right]$

$= \pi\left[\frac{9\pi}{8} + \frac{15}{4} - 2\sqrt{2}\right]$ units^3.

(d) $h(x) = \sin^{-1}x + \cos^{-1}x, \quad 0 \le x \le 1.$

(i) $h'(x) = \dfrac{1}{\sqrt{1-x^2}} - \dfrac{1}{\sqrt{1-x^2}}$

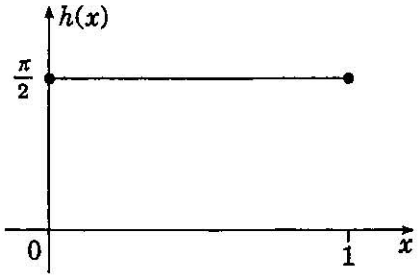
$= 0.$

(ii) Since $h'(x) = 0$, $h(x) = \text{const.}$

Now, choose $x = 0$.

When $x = 0$, $h(x) = 0 + \dfrac{\pi}{2} = \dfrac{\pi}{2}$

$\therefore \quad h(x) = \dfrac{\pi}{2}$ for $0 \le x \le 1.$



QUESTION FOUR

(a) $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} \qquad \begin{pmatrix}\sin\theta \neq 0\\ \cos\theta \neq 0\end{pmatrix}$

$= \dfrac{\sin 3\theta\cos\theta - \cos 3\theta\sin\theta}{\sin\theta\cos\theta}$

$= \dfrac{\sin(3\theta - \theta)}{\sin\theta\cos\theta}$

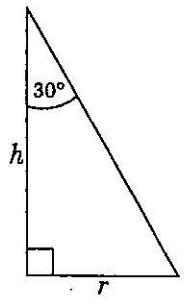
$\left[\text{using } \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta\right]$

$= \dfrac{\sin 2\theta}{\sin\theta\cos\theta}$

$= \dfrac{2\sin\theta\cos\theta}{\sin\theta\cos\theta}$

$= 2.$

(b)



(i) $\dfrac{r}{h} = \tan 30°$

$\therefore \dfrac{r}{h} = \dfrac{1}{\sqrt{3}}$

$\therefore \quad r = \dfrac{h}{\sqrt{3}}.$

(ii) $V = \dfrac{1}{3}\pi r^2 h$

$\therefore \quad V = \dfrac{1}{3}\pi\left(\dfrac{h}{\sqrt{3}}\right)^2 h$

$\therefore \quad V = \dfrac{\pi h^3}{9}.$

(iii) We want $\dfrac{dh}{dt}$ when $h = 3$,

given that $\dfrac{dV}{dt} = 0{\cdot}5 \text{ m}^3/\text{s}.$

Now $\dfrac{dV}{dt} = \dfrac{dV}{dh} \times \dfrac{dh}{dt}.$

But $\dfrac{dV}{dt} = 0{\cdot}5$

and $\dfrac{dV}{dh} = \dfrac{3\pi h^2}{9} = \dfrac{\pi h^2}{3}$

$\therefore \quad 0{\cdot}5 = \dfrac{\pi h^2}{3} \times \dfrac{dh}{dt}$

$\therefore \quad \dfrac{dh}{dt} = \dfrac{0{\cdot}5 \times 3}{\pi h^2}.$

When $h = 3$, $\dfrac{dh}{dt} = \dfrac{0{\cdot}5 \times 3}{\pi \times 3^2}.$

$\therefore$ Rate is $\dfrac{1}{6\pi}$ m/s.

(c) (i) When folded, $TX = PX = 15$ cm

$TY = RY = 20$ cm

$XY = AB = 25$ cm.

In ΔTXY, $TX^2 + TY^2 = 15^2 + 20^2$

$= 225 + 400$

$= 625$

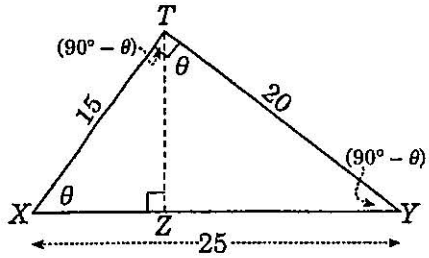
$= 25^2$

$= XY^2.$

$\therefore$ Pythagoras' theorem is satisfied.

$\therefore$ ΔTXY is right angled at T.

(ii)



$\Delta TXZ \parallel \Delta YTZ \parallel \Delta YXT$ (equiangular)

$\therefore \quad \dfrac{TZ}{TX} = \dfrac{YT}{YX}$ (corresponding sides in same ratio)

$\therefore \quad TZ = \dfrac{TX.YT}{YX}$

$= \dfrac{15.20}{25}$

$= 12$

$\therefore$ T is 12 cm above base.

OR Let $\angle TXY = \theta$,

then $\sin\theta = \dfrac{TY}{XY} = \dfrac{20}{25} = \dfrac{4}{5}.$

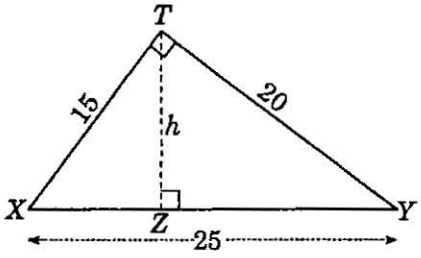
In ΔTZX, $\sin\theta = \dfrac{TZ}{XT} = \dfrac{TZ}{15}$

$\therefore \quad \dfrac{4}{5} = \dfrac{TZ}{15}$

$\therefore \quad TZ = \dfrac{4}{5} \times 15$

$\therefore \quad TZ = 12$ cm.

OR



Area $\Delta TXY = \dfrac{1}{2} \times 15 \times 20$

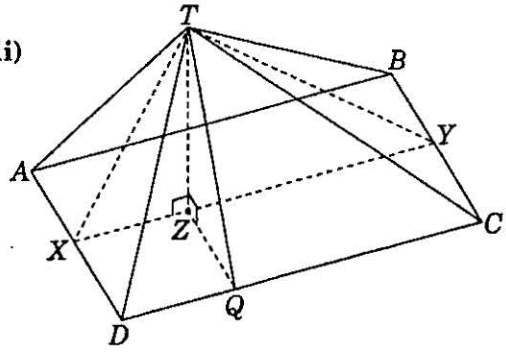
Also, area $\Delta TXY = \dfrac{1}{2} \times 25 \times h$

$\therefore \quad 15 \times 20 = 25 \times h$

$\therefore \quad h = \dfrac{15 \times 20}{25}$

$= 12$ cm.

(iii)



X, Y are midpoints of AD, BC respectively,

$\therefore \; XD = YC = ZQ = 5$ cm.

Angle between face DCT and base is given by $\angle TQZ$.

$$\tan \angle TQZ = \frac{TZ}{QZ} = \frac{12}{5}$$

$\therefore \quad \angle TQZ \approx 67° 23'$.

QUESTION FIVE

(a) $\dfrac{dT}{dt} = k\left(T - T_0\right), \quad \therefore \; T = T_0 + Ae^{kt}$.

(i) $T_0 = -20$

At $t = 0, \quad T = 100$

$100 = -20 + A$

$A = 120.$

At $t = 3, \quad T = 70$

$70 = -20 + 120e^{k \times 3}$

$e^{3k} = \dfrac{90}{120}$

$k = \dfrac{1}{3}\ln\dfrac{3}{4} \approx -0{\cdot}095\,89$

(ii) Let $t = 0$ when cup placed in environment at 20° C.

Then $T = 20 + Be^{kt}$.

At $t = 0, \quad T = 70$

$\therefore \quad 70 = 20 + Be^0$

$\therefore \quad B = 50$

$\therefore \quad T = 20 + 50e^{kt}$

where $k \approx -0{\cdot}095\,894\ldots$

At $t = 15, \; T = 20 + 50e^{15k}$

$\approx 31{\cdot}86°$.

(b) $v = \sqrt{10x - x^2}$

$\therefore \quad v^2 = 10x - x^2$

$\frac{1}{2}v^2 = 5x - \frac{1}{2}x^2$

$a = \dfrac{d}{dx}\left(\frac{1}{2}v^2\right) = 5 - x,$

$\therefore$ when $x = 4, \quad a = 1.$

(c) **(i)** Whatever exit the first mouse cor out, there is a $\frac{1}{5}$ probability that other mouse comes out that exit

$\therefore \; P(\text{all same exit}) = \left(\frac{1}{5}\right)^3 = \frac{1}{125}.$

(ii) $P(A, B, C$ same; D different)

$= \left(\frac{1}{5}\right)^2 \cdot \frac{4}{5} = \frac{4}{125}.$

(iii) There are 4 ways of choosing th mouse to be *different*.

$\therefore \; P(3$ same, 1 different)

$= \frac{4}{125} \times 4 = \frac{16}{125}.$

(iv) The possible ways of the mice coming out are:

- all same exit
- 3 same, 1 different
- no more than 2 out same exit

$\therefore \; P(\text{no more than 2 out same e}$

$= 1 - \big(P(\text{all same exit})$

$+ P(3 \text{ same, 1 different}$

$= 1 - \left(\frac{1}{125} + \frac{16}{125}\right)$

$= \frac{108}{125}.$

OR $P(\text{all different}) = \frac{4}{5}\times\frac{3}{5}\times\frac{2}{5} = \frac{24}{125}$

$P(2 \text{ same, others different})$

$= {}^4C_2 \times \frac{1}{5}\times\frac{4}{5}\times\frac{3}{5} = \frac{72}{125}$

$P(2 \text{ same, other 2 same})$

$= \frac{1}{2}\times {}^4C_2 \times \left(1\times\frac{1}{5}\right)\times\left(\frac{4}{5}\times\frac{1}{5}\right)$

$= \frac{12}{125}$

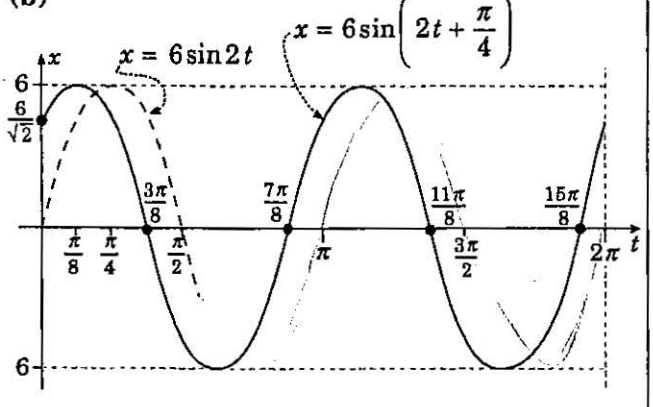
$\therefore$ total $= \frac{74+72+12}{125} = \frac{108}{125}$.

QUESTION SIX

$$x = 6\sin\left(2t + \frac{\pi}{4}\right)$$

(a) Period is $T = \frac{2\pi}{2} = \pi$. Amplitude is 6.

(b)



(c) $x = 6\sin\left(2t + \frac{\pi}{4}\right)$

$\frac{dx}{dt} = 6\times 2\cos\left(2t + \frac{\pi}{4}\right)$

$\therefore \quad v = 12\cos\left(2t + \frac{\pi}{4}\right)$.

At $t = 0$, $v = 12\cos\left(0 + \frac{\pi}{4}\right)$

$= \frac{12}{\sqrt{2}}$

$= 6\sqrt{2}$.

$\therefore$ Velocity is $6\sqrt{2}$ m/s.

(d) If $x = 3$, $\quad 3 = 6\sin\left(2t + \frac{\pi}{4}\right)$

$\therefore \sin\left(2t + \frac{\pi}{4}\right) = \frac{1}{2}$

$\therefore \quad 2t + \frac{\pi}{4} = \frac{\pi}{6}, \frac{5\pi}{6}, \cdots$

$\therefore \quad 2t = -\frac{\pi}{12}, \frac{7\pi}{12}, \cdots$

$\therefore \quad t = -\frac{\pi}{24}, \frac{7\pi}{24}, \cdots$

$\therefore$ First time is $\frac{7\pi}{24}$ sec.

(e) (i) $y = 6\sin\left(2t + \frac{\pi}{4}\right) + \sin 2t$

$\frac{dy}{dt} = 12\cos\left(2t + \frac{\pi}{4}\right) + 2\cos 2t$

$\frac{d^2y}{dt^2} = -24\sin\left(2t + \frac{\pi}{4}\right) - 4\sin 2t$

$= -4\left[6\sin\left(2t + \frac{\pi}{4}\right) + \sin 2t\right]$

$\therefore \quad \frac{d^2y}{dt^2} = -4y,$

that is, equation of acceleration is in the form $\ddot{y} = -n^2 y$.

$\therefore$ Motion is simple harmonic about O.

(ii) $y = 6\left(\sin 2t\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos 2t\right)$

$+ \sin 2t$

$= 3\sqrt{2}\sin 2t + 3\sqrt{2}\cos 2t + \sin 2t$

$= \left(3\sqrt{2}+1\right)\sin 2t + 3\sqrt{2}\cos 2t$

which may be written in the form

$A(\sin 2t\cos\alpha + \cos 2t\sin\alpha)$

$= A\sin(2t+\alpha) \qquad (A > 0)$

where $A^2 = \left(3\sqrt{2}+1\right)^2 + \left(3\sqrt{2}\right)^2$

$= \left(18 + 6\sqrt{2} + 1\right) + 18$

$= 37 + 6\sqrt{2}$

$\therefore \quad A = \sqrt{37+6\sqrt{2}}.$

$\therefore$ Amplitude $= \sqrt{37+6\sqrt{2}}$.

OR $\dot{y} = 0$ (for max. y)

$\therefore \quad 12\left(\cos 2t\cos\frac{\pi}{4} - \sin 2t\sin\frac{\pi}{4}\right)$

$+ 2\cos 2t = 0$

$6\sqrt{2}\cos 2t - 6\sqrt{2}\sin 2t$

$+ 2\cos 2t = 0$

$\cos 2t(3\sqrt{2}+1) = 3\sqrt{2}\sin 2t$

$\tan 2t = \frac{3\sqrt{2}+1}{3\sqrt{2}}$

$\therefore \quad t = 0{\cdot}4452\ldots,\ 2{\cdot}016\,014\ldots$

So, for amplitude, find y when $t = 0{\cdot}4452$.

$$y = 6\sin\left(2\times 0{\cdot}4452 + \frac{\pi}{4}\right) + \sin\left(2\times 0{\cdot}4452\right)$$

$$\approx 6{\cdot}744.$$

OR Now, for SHM, $v^2 = n^2\left(A^2 - x^2\right)$.

Here, let $x = y$ and $n = 2$,

$$\therefore\ A^2 = \frac{v^2}{4} + y^2$$

$$y = 6\sin\left(2t + \frac{\pi}{4}\right) + \sin 2t.$$

From (i),

$$v = \dot{y} = 12\cos\left(2t + \frac{\pi}{4}\right) + 2\cos 2t.$$

Let $t = 0$: $y = 6\sin\frac{\pi}{4} + 0$

$$= 3\sqrt{2}$$

and $v = 12\cos\frac{\pi}{4} + 2$

$$= 6\sqrt{2} + 2$$

$$\therefore\ A^2 = \frac{72 + 24\sqrt{2} + 4}{4} + 18$$

$$= 37 + 6\sqrt{2}$$

$$\therefore\ A = \sqrt{37 + 6\sqrt{2}}.$$

QUESTION SEVEN

(a) $(1+x)^4\,(1+x)^9 = (1+x)^{13}$

${}^{13}C_4$ is coeff. of term in x^4 on RHS,

$\therefore$ want terms in x^4 from LHS.

Now $(1+x)^4\,(1+x)^9$

$$= \left({}^4C_0 + {}^4C_1 x + {}^4C_2 x^2 + {}^4C_3 x^3 + {}^4C_4 x^4\right) \times \left({}^9C_0 + {}^9C_1 x + {}^9C_2 x^2 + {}^9C_3 x^3 + {}^9C_4 x^4 + \ldots\right)$$

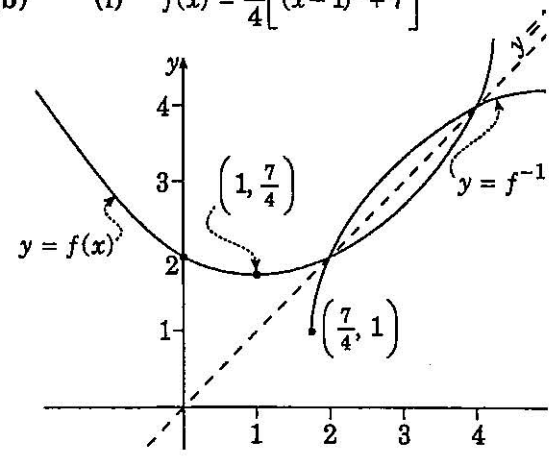
$\therefore$ x^4 term

$$= x^4\left({}^4C_0\,{}^9C_4 + {}^4C_1\,{}^9C_3 + {}^4C_2\,{}^9C_2 + {}^4C_3\,{}^9C_1 + {}^4C_4\,{}^9C_0\right).$$

$\therefore$ Equating coefficients:

$${}^4C_0\,.\,{}^9C_4 + {}^4C_1\,.\,{}^9C_3 + {}^4C_2\,.\,{}^9C_2 + {}^4C_3\,.\,{}^9C_1 + {}^4C_4\,.\,{}^9C_0 = {}^{13}C_4.$$

(b) **(i)** $f(x) = \frac{1}{4}\left|(x-1)^2 + 7\right|$



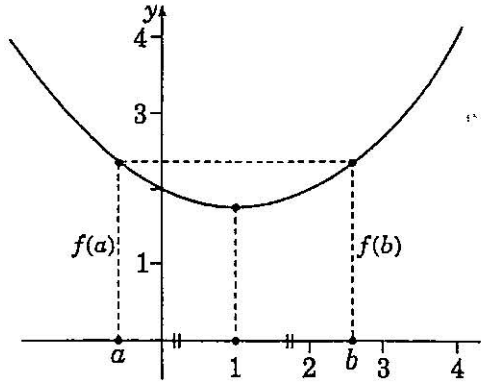
(ii) Largest domain containing $x = 3$ $f^{-1}(x)$ is $x \geq 1$ (horizontal line t

(iii) On graph: Reflect $y = f(x)$ in t line $y = x$ (for $x \geq 1$).

(iv) From graph, domain of inverse: x

(v) Now $f^{-1}\left(f(x)\right) = x$ if x is in the domain where f has an inverse.

Also, $f^{-1}\left(f(a)\right)$ exists if $f(a)$ is domain of $f^{-1}(x)$.



Let $a < 1$.

Now, from above graph, there is value, b, where $f(a) = f(b)$.

And, since $b > 1$, $f^{-1}\left(f(b)\right) = b$

$$\therefore\quad f^{-1}\left(f(a)\right) = b$$

Now, by symmetry, the midpoin between a and b is 1,

$$\therefore\quad \frac{a+b}{2} = 1$$

$$b = 2 - a$$

$$\therefore\ f^{-1}\left(f(a)\right) = 2 - a.$$

OR Algebraically:

$$f(x) = y = \frac{1}{4}\left[(x-1)^2+7\right]$$

$$\therefore f^{-1}(x):\ x = \frac{1}{4}\left[(y-1)^2+7\right]$$

$$4x = (y-1)^2+7$$

$$y = 1 \pm \sqrt{4x-7}.$$

But from graph the upper branch is the inverse for $x \geq 1$,

that is, $y = 1+\sqrt{4x-7}$.

$$f(a) = \frac{1}{4}\left[(a-1)^2+7\right]$$

$$\therefore f^{-1}\big(f(a)\big)$$

$$= 1+\sqrt{4\left[\frac{1}{4}(a-1)^2+7\right]-7}$$

$$= 1+\sqrt{(a-1)^2+7-7}$$

$$= 1+\sqrt{(a-1)^2}.$$

Now $\sqrt{(a-1)^2} = \begin{cases} -(a-1) & \text{if } a < 1 \\ a-1 & \text{if } a \geq 1 \end{cases}$

$$\therefore \quad f^{-1}\big(f(a)\big) = \begin{cases} 2-a & \text{if } a < 1 \\ a & \text{if } a \geq 1 \end{cases}$$

If a is not in domain from (ii), $a < 1$,

$$\therefore \quad f^{-1}\big(f(a)\big) = 2-a.$$

(vi) Since the graph of $y = f^{-1}(x)$ is obtained by reflecting $y = f(x)$ in the line $y = x$, the two curves intersect on this line.

$\therefore$ Find where $y = f(x)$ meets $y = x$.

$$\left.\begin{aligned} y &= \frac{1}{4}\left((x-1)^2+7\right) \\ y &= x \end{aligned}\right\}$$

$$\therefore \quad 4x = (x-1)^2+7$$

$$4x = x^2-2x+8$$

$$0 = x^2-6x+8$$

$$= (x-2)(x-4)$$

$$\therefore \quad x = 2, 4.$$

$\therefore$ Points are (2, 2), (4, 4).

OR From graph $y = f(x)$, seems to meet $y = x$ at (2, 2), (4, 4).

Test $x = 2$:

$$2 = \frac{1}{4}\left[(2-1)^2+7\right]$$

$$2 = \frac{1}{4} \times 8. \quad \text{TRUE}$$

Test $x = 4$:

$$4 = \frac{1}{4}\left[(4-1)^2+7\right]$$

$$2 = \frac{1}{4} \times 16. \quad \text{TRUE}$$

END OF 3 UNIT (ADDITIONAL) AND 3/4 UNIT (COMMON) MATHEMATICS SOLUTIONS