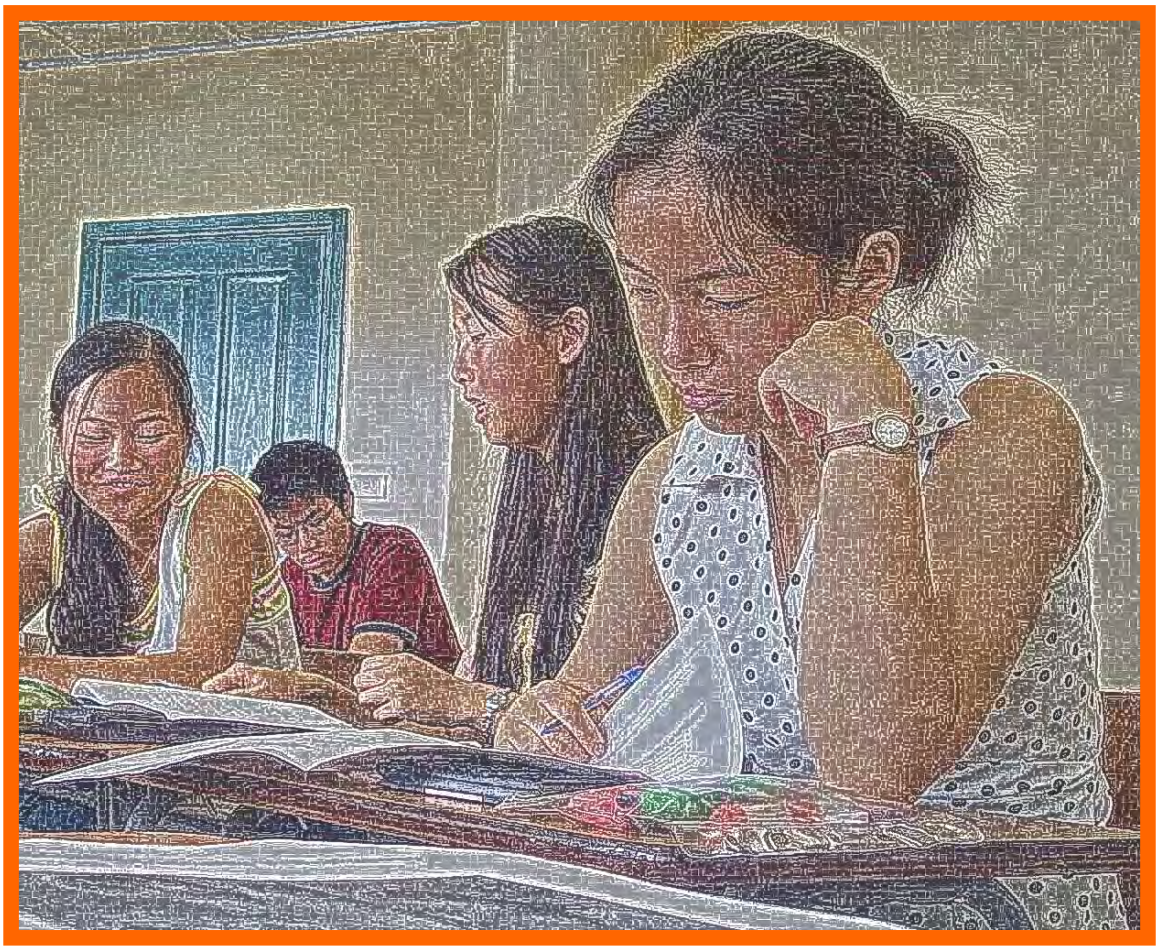
1996 HSC Full Solutions

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QUESTION 1

(a)

$P(x) = 2x^3 + x + a$

If (x-a) is a factor then P(a)=0

$P(2) = 2 \times 2^3 + 2 + a$

$0 = 18 + a$

$\therefore a = -18$

(b)

$y = -2x + 4 \therefore m_1 = -2$

$y = x - 2 \therefore m_2 = 1$

$Now \ \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$\tan\theta = \left|\frac{-2-1}{1+-2\times 1}\right|$

$= \left|\frac{-3}{-1}\right|$

$= 3$

$\therefore \theta = \tan^{-1} 3 \approx 72^0$

(c)

$A(-1,2)$ $B(3,5)$

$3 : -1$

$x = \frac{3.3 + -1.-1}{3 + -1}$ $y = \frac{3.5 + -1.2}{3 + -1}$

$x = 5$ $y = \frac{13}{2}$

$C : (5, \frac{13}{2})$

(d)

3 men from 8: ${}^8C_3$

4 women from 6: ${}^6C_4$

$\therefore Total\ ways = {}^8C_3 \times {}^6C_4$

(e)

$\frac{2}{x-1} \le 1$

$(x-1)^2 \times \frac{2}{x-1} \le 1 \times (x-1)^2$

$(x-1) \times 2 \le x^2 - 2x + 1$

$2x - 2 \le x^2 - 2x + 1$

$0 \le x^2 - 4x + 3$

$0 \le (x-1)(x-3)$

$\therefore x < 1 \ \ or \ \ x \ge 3$

(f)	Let u = e^x $\therefore du = e^x dx$ $\therefore \int \frac{e^x dx}{1+e^{2x}}$ $= \int \frac{du}{1+u^2} = \tan^{-1} u + C$ $= \tan^{-1} e^x + C$

Question 2

(a)(i)

$f(x) = x^3 - \ln(x+1)$

$f(0.8) = 0.8^3 - \ln(0.8+1) \approx -0.076 < 0$

$f(0.9) = 0.9^3 - \ln(0.9+1) \approx -0.087 > 0$

Since function changes from +ve to -ve then there is a root in that interval

(a)(ii)

Midpoint = 0.85

$f(0.85) = 0.85^3 - \ln(0.85+1) \approx -0.001 < 0$

Hence the root lies between 0.85 and 0.9

$\therefore root = 0.9$ correct to 1 decimal place

(b)

From the table of standard integrals use $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x+\sqrt{x^2+a^2}) + C$

$$\int \frac{dx}{\sqrt{x^2+8^2}} = \ln(x+\sqrt{x^2+8^2}) + C$$

$$\int_6^{15} \frac{dx}{\sqrt{x^2+8^2}} = \left[\ln(x+\sqrt{x^2+8^2})\right]_6^{15}$$

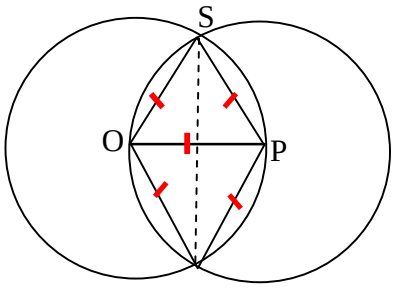
$$= \ln(15+\sqrt{15^2+8^2}) - \ln(6+\sqrt{6^2+8^2})$$

$$= \ln(15+17) - \ln(6+10)$$

$$= \ln(32) - \ln(16)$$

$$= \ln 2$$

(c) (i)



$radius = d$

$d = OP$ *in circle centre* O

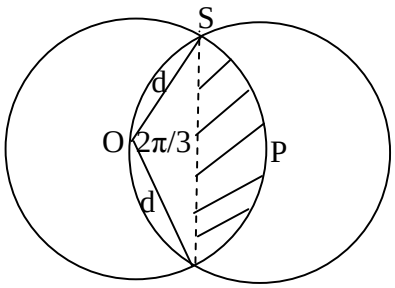
$\therefore OS = d$ $(= radius)$

$d = PO$ *in circle centre* P

$\therefore PS = d$ $(= radius)$

$\therefore OP = OS = PS$ $(all = d)$

$\therefore \Delta SOP$ *equilateral*

(c)(ii)	$\angle SOP = \frac{\pi}{3}$ (angles in an equilateral Δ) ΔTOP *also equilateral for same reasons in part* (*i*) $\therefore \angle TOP = \frac{\pi}{3}$ $\therefore \angle SOT = \angle TOP + \angle SOP = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$
(c)(iii)	Shaded area = $2 \times$ Area of segment SPT $\therefore 2 \times$ Area of segment SPT $= 2 \times$ (Area of sector OSPT - Area of triangle SOT) $= 2 \times \left(\frac{1}{2} r^2 \theta - \frac{1}{2} ab \sin C \right)$ $= 2 \times \left(\frac{1}{2} d^2 \frac{2\pi}{3} - \frac{1}{2} d.d \sin \frac{2\pi}{3} \right)$ $= 2 \times \left(d^2 \frac{\pi}{3} - \frac{d^2}{2} \sin \frac{2\pi}{3} \right)$ $= 2d^2 \left(\frac{\pi}{3} - \frac{1}{2} \frac{\sqrt{3}}{2} \right)$ $= 2d^2 \left(\frac{\pi}{3} - \frac{\sqrt{3}}{4} \right) cm^2$ 

Question 3

(a) (i)

Even functions: $f(x) = f(-x)$

$$f(x) = \frac{1}{\sqrt{16 - x^2}}$$

$$f(-x) = \frac{1}{\sqrt{16 - (-x)^2}}$$

$$= \frac{1}{\sqrt{16 - x^2}} = f(x)$$

$\therefore$ *even*

(a) (ii)

$$A = \int_{-2}^{2} \frac{1}{\sqrt{16 - x^2}}.dx$$

$$= 2\int_{0}^{2} \frac{1}{\sqrt{16 - x^2}}.dx$$

$$= 2\left[\sin^{-1}\frac{x}{4}\right]_0^2$$

$$= 2\left(\sin^{-1}\frac{2}{4}\right) - 2\left(\sin^{-1}\frac{0}{4}\right)$$

$$= 2 \times \frac{\pi}{6} = \frac{\pi}{3}\ units^2$$

(b)

from $\cos 2\alpha = 2\cos^2\alpha - 1$ we can

use $\cos^2\alpha = \frac{1}{2}(\cos 2\alpha - 1)$

$$\therefore \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 x dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2x)\,dx = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 + \cos 2x)\,dx$$

$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(\left(\frac{\pi}{2} + \frac{\sin 2\frac{\pi}{2}}{2}\right) - \left(\frac{\pi}{4} + \frac{\sin 2\frac{\pi}{4}}{2}\right)\right)$$

$$= \frac{1}{2}\left(\left(\frac{\pi}{2} + 0\right) - \left(\frac{\pi}{4} + \frac{1}{2}\right)\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi}{8} - \frac{1}{4}$$

(c)

$$V = \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(2+\cos x\right)^2 dx$$

$$= \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(4+4\cos x+\cos^2 x\right)dx$$

$$= \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(4+4\cos x\right)dx + \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\cos^2 x\right)dx$$

$$= \pi\left[4x+4\sin x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \pi\left(\frac{\pi}{8}-\frac{1}{4}\right) \quad \textit{from part (b)}$$

$$= \pi\left(4\times\frac{\pi}{2}+4\sin\frac{\pi}{2}\right)-\pi\left(4\times\frac{\pi}{4}+4\sin\frac{\pi}{4}\right)+\pi\left(\frac{\pi}{8}-\frac{1}{4}\right)$$

$$= \pi\left(2\pi+4\right)-\pi\left(\pi+\frac{4}{\sqrt{2}}\right)+\pi\left(\frac{\pi}{8}-\frac{1}{4}\right)$$

$$= \pi\left(2\pi+4-\pi-\frac{4}{\sqrt{2}}+\frac{\pi}{8}-\frac{1}{4}\right)$$

$$= \pi\left(\frac{15}{4}-2\sqrt{2}+\frac{9\pi}{8}\right)\textit{units}^3$$

(d) (i)

$$h(x) = \sin^{-1}x+\cos^{-1}x: \quad 0\le x\le 1$$

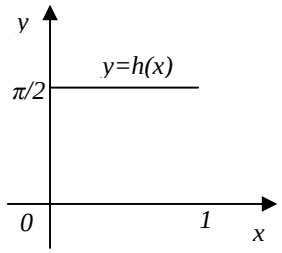
$$h'(x) = \frac{1}{\sqrt{1-x^2}}+\frac{-1}{\sqrt{1-x^2}} = 0$$

(d) (ii)

Since h'(x)=0 then it must be a horizontal line

$$h(0) = 0+\frac{\pi}{2} = \frac{\pi}{2}$$

$$\therefore h(x): y = \frac{\pi}{2}$$



Question 4

(a) (i)

$$LHS = \frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta}$$

$$= \frac{\sin 2\theta\cos\theta + \cos 2\theta\sin\theta}{\sin\theta} - \frac{\cos\theta\cos 2\theta - \sin\theta\sin 2\theta}{\cos\theta}$$

$$= \frac{\sin 2\theta\cos\theta}{\sin\theta} + \cos 2\theta - \cos 2\theta + \frac{\sin\theta\sin 2\theta}{\cos\theta}$$

$$= \sin 2\theta\left(\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}\right)$$

$$= \sin 2\theta\left(\frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}\right)$$

$$= \sin 2\theta\left(\frac{1}{\sin\theta\cos\theta}\right)$$

$$= 2\sin\theta\cos\theta\left(\frac{1}{\sin\theta\cos\theta}\right) = 2 = RHS$$

(b) (i)

Half the apex angle = 30^0

$$\therefore \tan 30^0 = \frac{r}{h} = \frac{1}{\sqrt{3}}$$

$$\therefore r = \frac{h}{\sqrt{3}}$$

(b) (ii)

$$V = \frac{1}{3}\pi r^2 h$$

$$Use\ r = \frac{h}{\sqrt{3}}$$

$$\therefore V = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h = \frac{\pi h^3}{9}$$

(b) (iii)

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$Now\ V = \frac{\pi h^3}{9} \therefore \frac{dV}{dh} = \frac{\pi h^2}{3}$$

$$\therefore \frac{dh}{dV} = \frac{3}{\pi h^2}$$

$$Given \quad \frac{dV}{dt} = 0.5$$

$$\therefore \frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$= \frac{3}{\pi h^2} \times 0.5$$

$$(h = 3) := \frac{3}{\pi 3^2} \times 0.5$$

$$= \frac{1}{6\pi}\, ms^{-1}$$

(c)(i)

(c)(ii)	
(c)(iii)	

Question 5

(a) (i)

$$\frac{dT}{dt} = k(T - T_0)$$

$$\therefore T = T_0 + Ae^{kt}$$

At $t = 0$ T=100^0 $T_0 = -20$

$$\therefore 100 = -20 + Ae^0$$

$$\therefore A = 120$$

$$\therefore T = -20 + 120e^{kt}$$

At $t = 3$ $T = 70^0$

$$\therefore 70 = -20 + 120e^{3k}$$

$$90 = 120e^{3k}$$

$$e^{3k} = \frac{90}{120} = 0.75$$

$$3k = \ln 0.75$$

$$k = \frac{\ln 0.75}{3} \approx -0.0959$$

(a) (ii)

Given $T_0 = 20, T = 70, t = 0$

$$\therefore 70 = 20 + Ae^0$$

$$\therefore A = 50$$

$$\therefore T = 20 + 50e^{-0.0959t}$$

At $t = 15$

$$T = 20 + 50e^{-0.0959 \times 15}$$

$$T \approx 32^0 C$$

(b)

$$v = \sqrt{10x - x^2}$$

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$a = \frac{d}{dx}\left(\frac{1}{2}\sqrt{10x - x^2}^2\right)$$

$$= \frac{d}{dx}\left(\frac{1}{2}\left(10x - x^2\right)\right)$$

$$= \frac{1}{2}\left(10 - 2x\right)$$

$$= 5 - x$$

$$at\ x = 4$$

$$a = 5 - 4 = 1\, ms^{-2}$$

(c)(i)

Let the first mouse come out any exit. Then the probability that B will come out there is 1/5

Cis 1/5 and D is 1/5

$$\text{P(all the same exit)} = \frac{1}{5} \times \frac{1}{5} \times \frac{1}{5} = \frac{1}{125}$$

(c)(ii)	Let A come out any exit. B comes out the same one is 1/5 C comes out the same one is 1/5. Dcomes out a different one is 4/5 P(Event)= $\frac{1}{5}\times\frac{1}{5}\times\frac{4}{5}=\frac{4}{125}$
(c)(iii)	There are 4 possibilities: A B C the same exit & D a different exit A B D the same exit & C a different exit A D C the same exit & B a different exit D B C the same exit & A a different exit $P(E)=4\times\frac{1}{5}\times\frac{1}{5}\times\frac{4}{5}=\frac{16}{125}$
(c) (iv)	The possibilities of more than two mice from the same exit: All 4 same exit $=\frac{1}{125}$ All 3 same exit 1 different $=\frac{16}{125}$ $P(\text{NO more than 2})=1-\left(\frac{16}{125}+\frac{1}{125}\right)=1-\frac{17}{125}=\frac{108}{125}$

Question 6

(a)

$P = \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$

$Amplitude = 6$

(b)

Find when $2t + \frac{\pi}{4} = 0$

$2t + \frac{\pi}{4} = 0$

$2t = -\frac{\pi}{4}$

$t = -\frac{\pi}{8}$

Now it crosses the x axis twice in each period of π.

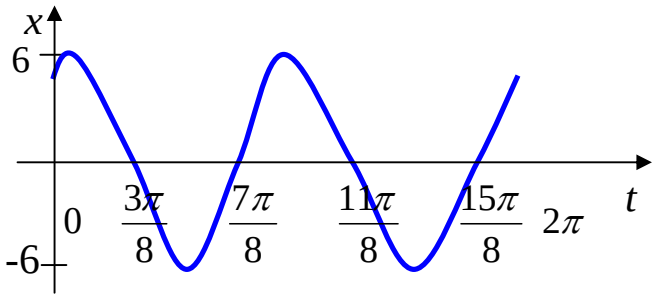
i.e. every multiple of $\frac{\pi}{2}$ $-\frac{\pi}{8}$

$\therefore t = -\frac{\pi}{8}, \frac{\pi}{2} - \frac{\pi}{8}, \pi - \frac{\pi}{8}, \frac{3\pi}{2} - \frac{\pi}{8}, 2\pi - \frac{\pi}{8} \ldots$

$\therefore t = -\frac{\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}, \frac{19\pi}{8}, \ldots$

However for $0 \le t \le 2\pi$

$t = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$



(c)

$x = 6\sin(2t + \frac{\pi}{4})$

$\frac{dx}{dt} = 12\cos(2t + \frac{\pi}{4})$

when $t = 0$

$\frac{dx}{dt} = 12\cos(0 + \frac{\pi}{4})$

$= 12 \times \frac{1}{\sqrt{2}} = \frac{12}{\sqrt{2}} = 6\sqrt{2}\ ms^{-1}$

(d)	$6\sin(2t+\frac{\pi}{4})=3$ $\sin(2t+\frac{\pi}{4})=\frac{1}{2}$ $(2t+\frac{\pi}{4})=\pi-\frac{\pi}{6}$ $2t=\pi-\frac{\pi}{6}-\frac{\pi}{4}=\frac{7\pi}{12}$ $t=\frac{7\pi}{24}\,s$
(e) (i)	
(e) (ii)	

Question 7

(a)

$(1+x)^4(1+x)^9 = (1+x)^{13}$

The coefficient of x^4 on the LHS is

$= {}^4C_0\,{}^9C_4 + {}^4C_1\,{}^9C_3 + {}^4C_2\,{}^9C_2 + {}^4C_3\,{}^9C_1 + {}^4C_4\,{}^9C_0$

The coefficient of x^4 on the RHS is

$= {}^{13}C_4$

$\therefore {}^4C_0\,{}^9C_4 + {}^4C_1\,{}^9C_3 + {}^4C_2\,{}^9C_2 + {}^4C_3\,{}^9C_1 + {}^4C_4\,{}^9C_0 = {}^{13}C_4$

(b) (i)

$y = \frac{1}{4}\left[(x-1)^2 + 7\right]$

Minimum @ $(1, \frac{7}{4})$

$x = 0; y = 2$

(b) (ii)

Since vertex @ $x = 1$ *then* $x \geq 1$ *or* $x \leq 1$ *would be the domain for* $f^{-1}(x)$ *to be a function. However to contain* $x = 3$ *then the domain would be* $x \geq 1$.

(b) (iii) See graph

(b) (iv)

The domain of $f^{-1}(x)$ *is* $x \geq \frac{7}{4}$

(b) (v)

$f : y = \frac{1}{4}\left[(x-1)^2 + 7\right]$

$4y = (x-1)^2 + 7$

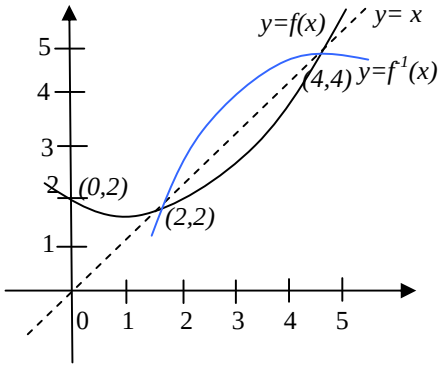
$4y - 7 = (x-1)^2$

$(x-1)^2 = 4y - 7$

$x - 1 = \pm\sqrt{4y-7}$

$x = 1 \pm \sqrt{4y-7}$

$f^{-1} : y = 1 \pm \sqrt{4x-7}$



Now: $f^{-1}(f(a)) = f^{-1}(\frac{1}{4}\left[(a-1)^2 + 7\right])$

$= 1 \pm \sqrt{4 \times \frac{1}{4}\left[(a-1)^2 + 7\right] - 7}$

$= 1 \pm \sqrt{\left[(a-1)^2 + 7\right] - 7}$

$= 1 \pm \sqrt{(a-1)^2}$

$= 1 \pm \sqrt{(a-1)^2}$

$= 1 \pm (a-1) = a \text{ or } 2-a$

Since a is not in the domain given in part (ii) *then*

$a < 1$ *and* $f^{-1}(f(a)) = 2 - a$

T

(b) (vi)	The points of intersection lie on the line y=x $\therefore solve: y = \frac{1}{4}\left[(x-1)^2+7\right]$ & $y = x$ simultaneously $\therefore x = \frac{1}{4}\left[(x-1)^2+7\right]$ $4x = (x-1)^2+7$ $4x-7 = (x-1)^2$ $4x-7 = x^2-2x+1$ $0 = x^2-6x+8$ $0 = (x-2)(x-4)$ $\therefore (2,2)$ & $(4,4)$