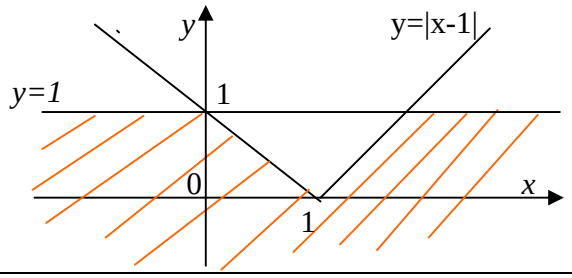
1995 HSC Full Solutions

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QUESTION 1

(a)



(b)

$$xy = 1$$

$$y = \frac{1}{x}$$

$$\int_1^4 \frac{1}{x}\,dx = \left[\ln x\right]_1^4 = \ln 4 - \ln 1 = \ln 4$$

(c)

$$\lim_{x\to 0}\frac{\sin x}{5x} = \frac{1}{5}\lim_{x\to 0}\frac{\sin x}{x} = \frac{1}{5}\times 1 = \frac{1}{5}$$

(d)

$$2^{n+1} + 2^n = 2^1 \times 2^n + 2^n$$

$$= 2^n(2+1)$$

$$= 3.2^n$$

$$\frac{2^{1001} + 2^{1000}}{3} = \frac{3\times 2^{1000}}{3}$$

$$= 2^{1000}$$

(e)

$$Let\ u = 9 - x^2$$

$$\therefore du = -2x\,dx$$

$$When\ x = 0, u = 9 \quad When\ x = 1, u = 8$$

$$\therefore \int_0^1 6x\sqrt{9-x^2}\,dx = -3\int_0^1 -2x\sqrt{9-x^2}\,dx$$

$$= -3\int_9^8 \sqrt{u}\,du = -3\int_9^8 u^{\frac{1}{2}}\,du$$

$$= -3\left[\frac{2}{3}u^{\frac{3}{2}}\right]_9^8$$

$$= -2\left[u\sqrt{u}\right]_9^8$$

$$= -2(8\sqrt{8} - 9\sqrt{9}) = -2(16\sqrt{2} - 27)$$

$$= -32\sqrt{2} + 54 = 54 - 32\sqrt{2}$$

Question 2

(a) (i)

$f(x) = x^3 + 5x^2 + 17x - 10$

$f(0) = 0 + 0 + 0 - 10 = -10 < 0$

$f(2) = 2^3 + 5 \times 2^2 + 17 \times 2 - 10 = 52 > 0$

Crosses the x axis, has a root, between 0 & 2 because it changes from -ve to +ve

(a)(ii)

Try half way bewtween 0 and 2 i.e.x=1

$f(x) = x^3 + 5x^2 + 17x - 10$

$f(1) = 1 + 5 + 17 - 10 = 13$

$13 > 0$

At x=1 the curve is above the x axis

$\therefore$ *the* root lies between 0 and 1

so x=1 is a closer estimate

(a)(iii)

Try half way between 0 & 1 ie. x=$\frac{1}{2}$

$f(x) = x^3 + 5x^2 + 17x - 10$

$f(\frac{1}{2}) = \frac{1}{2}^3 + 5 \times \frac{1}{2}^2 + 17 \times \frac{1}{2} - 10 = -0.125$

So the root lies between x=$\frac{1}{2}$ and x=1

$\therefore$ the root is closer to x=1 then x=0

(b)

The area below y=3 forms a cylinder when rotated about the y axis

The radius, r =1 and the height h=3

$\therefore V = \pi r^2 h$

$V = \pi 1^2 3 = 3\pi$

The area rotated above the line $y = 3$ is:

$$V = \pi \int_3^6 x^2 dy = \pi \int_3^6 \frac{9}{y^2} dy = 9\pi \int_3^6 y^{-2} dy$$

$$= 9\pi \left[-y^{-1} \right]_3^6$$

$$= 9\pi \left[-\frac{1}{y} \right]_3^6$$

$$= 9\pi \left(-\frac{1}{6} - -\frac{1}{3} \right)$$

$$= 9\pi \times \frac{1}{6} = \frac{3\pi}{2} units^3$$

$$Total\ Volume = 3\pi + \frac{3\pi}{2} units^3$$

$$= \frac{6\pi}{2} + \frac{3\pi}{2} = \frac{9\pi}{2} units^3$$

(c)	$x^3 + 6x^2 - x - 30 = 0$ *Let* the roots be $\alpha, \beta, \& (\alpha + \beta)$ $\alpha + \beta + \alpha + \beta = -6$ $2\alpha + 2\beta = -6$ $\alpha + \beta = -3$ ...(1) $\alpha\beta(\alpha + \beta) = 30$ $\alpha\beta(-3) = 30$ $\alpha\beta = -10$ ...(2) from (1)$\alpha = -3 - \beta$ sub into (2) $(-3 - \beta)\ \beta = -10$ $-3\beta - \beta^2 = -10$ $3\beta + \beta^2 = 10$ $\beta^2 + 3\beta - 10 = 0$ $(\beta - 2)(\beta + 5) = 0$ $\therefore \beta = 2$ *or* $\beta = -5$ The three roots are 2,-5 and 2-5 i.e. 2,-3, & -5

Question 3

(a) (i)

$8\times 8\times 8 = 8^3 = 512$

(a) (ii)

$8\times 7\times 5 \text{ (or } {}^8P_3) = 336$

(a) (iii)

${}^8C_3 = 56$

(b)

${}^6C_r \left(x^2\right)^{6-r}\left(\frac{3}{x}\right)^r = {}^6C_r \times 3^r \times x^{12-2r-r}$

We want $12-3r=0$

$\therefore r = 4$

The term independent of x is ${}^6C_4 \times 3^4 = 1215$

(c) (i)

$\tan\beta = m$

$m = \frac{dy}{dx} = \frac{dy/dp}{dx/dp} = \frac{2a}{2} = a$

(c) (ii)

$m_{SP} = \frac{a^2-1}{2a}$

$m = \frac{1}{2}\left(\frac{a^2-1}{a}\right)$

$m = \frac{1}{2}\left(a-\frac{1}{a}\right)$

(c) (iii)

$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$

$\tan\theta = \left|\frac{a-\frac{1}{2}\left(a-\frac{1}{a}\right)}{1+a\times\frac{1}{2}\left(a-\frac{1}{a}\right)}\right|$

$= \left|\frac{\frac{1}{2}\left(a+\frac{1}{a}\right)}{\frac{1}{2}\left(a^2+1\right)}\right|$

$= \frac{a^2+1}{a(a^2+1)}$

$= \frac{1}{a}$

(c) (iv)

$\tan\left(\theta+\beta\right) = \frac{\tan\theta+\tan\beta}{1-\tan\theta\tan\beta}$

$= \frac{\frac{1}{a}+a}{1-\frac{1}{a}\times a} \to \infty$

$\therefore \theta+\beta = \frac{\pi}{2}$ since both angles acute

(c) (v)	If $\theta = \beta$ then $2\beta = \frac{\pi}{2}$ $\therefore \beta = \frac{\pi}{4}$ $a = \tan\frac{\pi}{4} = 1, \quad \therefore P(2,1)$

Question 4

(a)

$$f(x)=\frac{e^x}{3+e^x}$$

$$f'(x)=\frac{\left(3+e^x\right)e^x-e^x\left(e^x\right)}{\left(3+e^x\right)^2}$$

$$=\frac{3e^x+e^{2x}-e^{2x}}{\left(3+e^x\right)^2}$$

$$=\frac{3e^x}{\left(3+e^x\right)^2}>0 \text{ for all } x$$

(b)

$$f''(x)=\frac{3e^x(3-e^x)}{\left(3+e^x\right)^3}$$

$$Let\ 0=\frac{3e^x(3-e^x)}{\left(3+e^x\right)^3}$$

$$\therefore 0=3e^x(3-e^x)$$

$$\therefore 0=(3-e^x)$$

$$\therefore e^x=3$$

$$\therefore x=\ln 3$$

$$y=\frac{e^x}{\left(3+e^x\right)}=\frac{e^{\ln 3}}{\left(3+e^{\ln 3}\right)}$$

$$=\frac{3}{(3+3)}=\frac{1}{2}$$

$$\therefore \text{Inflection point} = (ln3,\frac{1}{2})$$

(c)

$f(x)>0$ *for all* x

since $e^x>0$ *for all* x

$$e^x<3+e^x$$

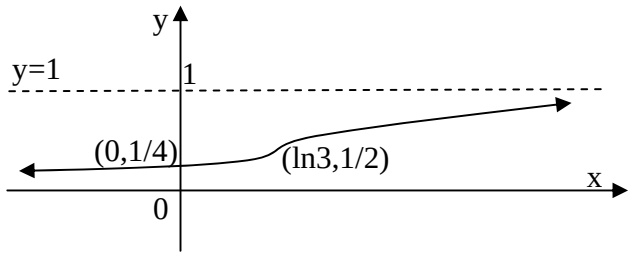
$$\therefore \frac{e^x}{3+e^x}<1$$

$$\therefore 0<\frac{e^x}{3+e^x}<1$$

(d)

$$\lim_{x\to\infty}\frac{e^x}{3+e^x}=\lim_{x\to\infty}\frac{e^x\times\frac{1}{e^x}}{3\times\frac{1}{e^x}+e^x\times\frac{1}{e^x}}=\lim_{x\to\infty}\frac{1}{\frac{3}{e^x}+1}=\frac{1}{1+0}=1$$

$$\lim_{x\to-\infty}\frac{e^x}{3+e^x}=\frac{0}{3+0}=0$$

(e)	
(f)	$f(x)$ has an inverse because for each value of x it has only one value for y. Also it is monotonic increasing over its domain. Also it passes the horizontal line test
(g)	$x = \frac{e^y}{3+e^y}$ $x(3+e^y) = e^y$ $3x + xe^y = e^y$ $3x = e^y - xe^y$ $3x = e^y(1-x)$ $\frac{3x}{1-x} = e^y$ $e^y = \frac{3x}{1-x}$ $\ln e^y = \ln\left(\frac{3x}{1-x}\right)$ $y = \ln\left(\frac{3x}{1-x}\right)$

Question 5

(a) (i)

$$\sin 2x = 2\sin^2 x \quad 0 < x < \pi$$

$$2\sin x\cos x = 2\sin^2 x$$

$$\sin x\cos x = \sin^2 x$$

$$0 = \sin x\cos x - \sin^2 x$$

$$0 = \sin x(\cos x - \sin x)$$

$$0 = \sin x \text{ (no solution } 0 < x < \pi)$$

$$\cos x - \sin x = 0$$

$$\cos x = \sin x$$

$$1 = \frac{\sin x}{\cos x}$$

$$\tan x = 1$$

$$x = \frac{\pi}{4}$$

(a) (ii)

$$0 < x < \frac{\pi}{4} \quad \tan x < 1$$

$$\therefore \frac{\sin x}{\cos x} < 1 \ \& \sin x > 0 \ \& \cos x > 0$$

$$\therefore \sin x < \cos x$$

$$\therefore \sin^2 x < \sin x\cos x$$

$$\therefore 2\sin^2 x < 2\sin x\cos x$$

$$\therefore 2\sin^2 x < \sin 2x$$

$$i.e. \sin 2x > 2\sin^2 x$$

(a) (iii)

$$A = \int_0^{\frac{\pi}{4}} \left(\sin 2x - 2\sin^2 x\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\sin 2x - 2.\frac{1}{2}(1-\cos 2x)\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\sin 2x - 1 + \cos 2x\right)dx$$

$$= \left[-\frac{1}{2}\cos 2x - x + \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$$

$$= \left(-\frac{1}{2}\cos 2\times\frac{\pi}{4} - \frac{\pi}{4} + \frac{1}{2}\sin 2\times\frac{\pi}{4}\right) - \left(-\frac{1}{2}\cos 2\times 0 - 0 + \frac{1}{2}\sin 2\times 0\right)$$

$$= \left(0 - \frac{\pi}{4} + \frac{1}{2}\right) - \left(-\frac{1}{2}.1 - 0 + 0\right)$$

$$= 1 - \frac{\pi}{4} \, units^3$$

(b) (i)	$p = 0.015 \ \therefore q = 1 - 0.015 = 0.985$ Probability of exactly one jackpot prize $= {}^{10}C_1 (0.985)^9 (0.015)^1$ ≈ 0.1309
(b) (ii)	$P(at\ least\ one\ \ jackpot) = 1 - P(no\ \ jackpots)$ $= 1 - 0.985^{10} \approx 0.1403$
(b)(iii)	To exceed \$200 000 it must make it to the $\frac{200\,000}{8000} = 25th\ game$ The probability of 25 games without a win $=0.985^{25} \approx 0.0653$

Question 6

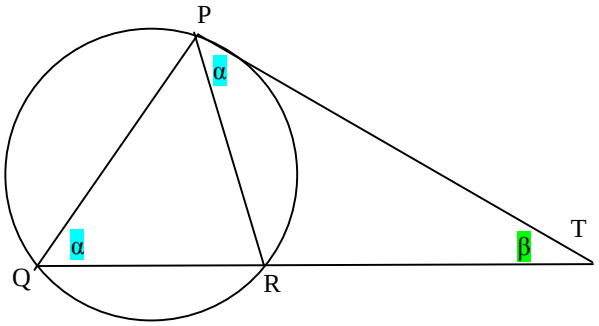
(a)(i)

In triangles PRT and QPT

$\angle$TPR=$\angle$RQP (angle in the alt. segment)

$\angle$T in common

$\therefore \Delta PRT|||\Delta QPT$ (AA)



(a) (ii)

$\frac{QT}{PT}=\frac{PT}{RT}$ (corresponding $\angle$'s on $|||\Delta$'s)

$\therefore PT^2 = QT \times RT$

(b) (i)

$\cos\theta = \frac{OP}{OT} = \frac{3}{x}$

(b) (ii)

$l = r\angle POQ$

$l = 3(\pi - \theta)$

$l = 3(\pi - \cos^{-1}\frac{x}{3})$

(b) (iii)

$l = 3\pi - 3\cos^{-1}\frac{3}{x}$

$$\frac{dl}{dx} = -3\left(-\frac{1}{\sqrt{1-\left(\frac{3}{x}\right)^2}}\times\frac{d}{dx}\left(\frac{3}{x}\right)\right) = -3\left(\frac{-1}{\sqrt{1-\frac{9}{x^2}}}\times\frac{d}{dx}3x^{-1}\right)$$

$$= -3\left(\frac{-1}{\sqrt{\frac{x^2-9}{x^2}}}\times -3x^{-2}\right) = \frac{-9}{\sqrt{\frac{x^2-9}{x^2}}}\times\frac{1}{x^2}$$

$$= \frac{-9}{\frac{\sqrt{x^2-9}}{x}}\times\frac{1}{x^2} = \frac{-9}{x\sqrt{x^2-9}}$$

Since x>0 then the significance of $\frac{dl}{dx}$ being negative

is that l decreases x increases and as l increases as x decreases.

(b) (iv)

$PT^2 = 3^2 - x^2 = 9 - x^2$

$\therefore PT = \sqrt{9-x^2}$

$l = 3\left[\pi - \cos^{-1}\frac{3}{x}\right]$

$\therefore s = 3\left[\pi - \cos^{-1}\frac{3}{x}\right] + \sqrt{9-x^2}$

(b) (v)

$$\frac{ds}{dt} = \frac{ds}{dx} \times \frac{dx}{dt}$$

$$= \frac{d}{dx}\left(3\left[\pi - \cos^{-1}\frac{3}{x}\right] + \sqrt{x^2-9}\right) \times \frac{dx}{dt}$$

$$= \frac{d}{dx}\left(3\pi - 3\cos^{-1}\frac{3}{x} + \left(x^2-9\right)^{\frac{1}{2}}\right) \times \frac{dx}{dt}$$

$$= \left(\frac{-9}{x\sqrt{x^2-9}} + \frac{1}{2}\left(x^2-9\right)^{-\frac{1}{2}} \times 2x\right) \times \frac{dx}{dt} \quad \textit{from part (iii)}$$

$$= \left(\frac{-9}{x\sqrt{x^2-9}} + \frac{x}{\sqrt{x^2-9}}\right) \times \frac{dx}{dt}$$

Now when x=10 and $\frac{dx}{dt} = 2$ then:

$$= \left(\frac{-9}{10\sqrt{10^2-9}} + \frac{10}{\sqrt{10^2-9}}\right) \times 2$$

$$= \left(\frac{-9}{10\sqrt{91}} + \frac{10}{\sqrt{91}}\right) \times 2$$

$$= \left(\frac{-9+100}{10\sqrt{91}}\right) \times 2$$

$$= \frac{91 \times 2}{10\sqrt{91}} = \frac{\sqrt{91}}{5} ms^{-1}$$

Question 7

(a)

Intitial conditions:

$At\ t=0: \dot{x}=V\cos\alpha$

$At\ t=0: \dot{y}=V\sin\alpha$

In the horizontal plane

$\ddot{x}=0$

$\dot{x}=C_1$

$At\ t=0: \dot{x}=V\cos\alpha$

$C_1=V\cos\alpha$

$\dot{x}=V\cos\alpha$

$x=Vt\cos\alpha+C_2$

$C_2=0$

$\therefore x=Vt\cos\alpha$

In the vertical plane:

$\ddot{y}=-g$

$\dot{y}=-gt+C_3$

$At\ t=0: \dot{y}=V\sin\alpha$

$C_3=V\cos\alpha$

$\therefore \dot{y}=-gt+V\sin\alpha$

$y=-\frac{1}{2}gt^2+Vt\sin\alpha+C_4$

$at\ t=0\ y=h$

$\therefore C_4=h$

$\therefore y=-\frac{1}{2}gt^2+Vt\sin\alpha+h$

(b)

Now from $x=Vt\cos\alpha \Rightarrow t=\frac{x}{V\cos\alpha}$ sub into

$y=-\frac{1}{2}g\left(\frac{x}{V\cos\alpha}\right)^2+V\frac{x}{V\cos\alpha}\sin\alpha+h$

$y=\frac{-gx^2}{2V^2\cos^2\alpha}+\frac{x}{\cos\alpha}\sin\alpha+h$

$y=\frac{-gx^2}{2V^2\cos^2\alpha}+\frac{x\sin\alpha}{\cos\alpha}+h$

$y=x^2\frac{-g}{2V^2\cos^2\alpha}+x\tan\alpha+h$

$y=h+x\tan\alpha-x^2\frac{g}{2V^2\cos^2\alpha}$

(c)	The ball clears the fence then at x=R, y>h $\therefore h + R\tan\alpha - R^2\dfrac{g}{2V^2\cos^2\alpha} \geq h$ $R\tan\alpha - R^2\dfrac{g}{2V^2\cos^2\alpha} \geq 0$ $R\left(\tan\alpha - R\dfrac{g}{2V^2\cos^2\alpha}\right) \geq 0$ $(\div \text{R which is positive \&} \neq 0)$ $\tan\alpha - R\dfrac{g}{2V^2\cos^2\alpha} \geq 0$ $\tan\alpha \geq R\dfrac{g}{2V^2\cos^2\alpha}$ $2V^2\cos^2\alpha\tan\alpha \geq Rg$ $(\times 2V^2\cos^2\alpha \text{ which is positive \&} \neq 0)$ $V^2 \geq \dfrac{Rg}{2\cos^2\alpha\tan\alpha}$ $\left(\cos^2\alpha\tan\alpha = \cos^2\alpha\dfrac{\sin\alpha}{\cos\alpha} = \sin\alpha\cos\alpha\right)$ $V^2 \geq \dfrac{gR}{2\sin\alpha\cos\alpha}$

(d)

If the ball hits cap C then when $x = r + R$ $y = 0$

and $V^2 \geq \dfrac{gR}{2\sin\alpha\cos\alpha}$

Now $y = h + x\tan\alpha - x^2\dfrac{g}{2V^2\cos^2\alpha}$

$\therefore 0 = h + (r+R)\tan\alpha - (r+R)^2\dfrac{g}{2V^2\cos^2\alpha}$

$0 = \dfrac{h}{(r+R)^2} + \dfrac{(r+R)\tan\alpha}{(r+R)^2} - \dfrac{(r+R)^2 g}{(r+R)^2 2V^2\cos^2\alpha}$

$0 = \dfrac{h}{(r+R)^2} + \dfrac{\tan\alpha}{(r+R)} - \dfrac{g}{2V^2\cos^2\alpha}$

$\dfrac{g}{2V^2\cos^2\alpha} = \dfrac{h}{(r+R)^2} + \dfrac{\tan\alpha}{(r+R)}$

$\dfrac{g}{2\cos^2\alpha} = \left[\dfrac{h}{(r+R)^2} + \dfrac{\tan\alpha}{(r+R)}\right]V^2$

Now since $V^2 \geq \dfrac{gR}{2\sin\alpha\cos\alpha}$

$\dfrac{g}{2\cos^2\alpha} \geq \left[\dfrac{h}{(r+R)^2} + \dfrac{\tan\alpha}{(r+R)}\right]\dfrac{gR}{2\sin\alpha\cos\alpha}$

$\dfrac{2\sin\alpha\cos\alpha}{2\cos^2\alpha} \geq \left[\dfrac{h}{(r+R)^2} + \dfrac{\tan\alpha}{(r+R)}\right]R$

$\tan\alpha \geq \left[\dfrac{h}{(r+R)^2} + \dfrac{\tan\alpha}{(r+R)}\right]R$

$\tan\alpha \geq \dfrac{Rh}{(r+R)^2} + \dfrac{R\tan\alpha}{(r+R)}$

$(r+R)^2\tan\alpha \geq Rh + R\tan\alpha(r+R)$

$(r+R)^2\tan\alpha - R\tan\alpha(r+R) \geq Rh$

$\tan\alpha\left((r+R)^2 - R(r+R)\right) \geq Rh$

$\tan\alpha\left(r^2 + 2rR + R^2 - Rr - R^2\right) \geq Rh$

$\tan\alpha\left(r^2 + rR\right) \geq Rh$

$\tan\alpha \geq \dfrac{Rh}{(r^2 + rR)}$

$\tan\alpha \geq \dfrac{Rh}{(r+R)r}$

(e)	If $V \le 50$ then $V^2 \le 2500$ $From\ (c)\ V^2 \ge \dfrac{gR}{2\sin\alpha\cos\alpha}$ $\therefore 2500 \ge \dfrac{gR}{2\sin\alpha\cos\alpha}$ $2\sin\alpha\cos\alpha \ge \dfrac{gR}{2500}$ $\sin 2\alpha \ge \dfrac{gR}{2500}$ Now g=10 R=80 $\therefore \sin 2\alpha \ge \dfrac{10\times 80}{2500}$ $\sin 2\alpha \ge \dfrac{8}{25}$ $\therefore 18.6^0 \le 2\alpha \le 161.3^0$ $9.3^0 \le \alpha \le 80.7^0$ Now it will be closest to the fence on its steepest trajectory $\therefore \alpha = 80.7^0$ $Now\ \tan\alpha \ge \dfrac{Rh}{(r+R)r}$ Using R=80 & h=1 and r=distance from the fence $\therefore \tan 80.7^0 \ge \dfrac{80\times 1}{(r+80)r}$ $6.09(r+80)r \ge 80$ $(r+80)r \ge \dfrac{80}{6.09}$ $r^2+80r \ge 13.14$ $r^2+80r-13.14 \ge 0$ The minimum positive value for r : $r = \dfrac{-80 \pm \sqrt{80^2 - 4\times 13.14}}{2}$ $r \approx 0.16m$ The closest point is $0.16m$ from the fence or $80.16m$ from the point O