1994 HIGHER SCHOOL CERTIFICATE SOLUTIONS 3 UNIT (ADDITIONAL) AND 3/4 UNIT (COMMON) MATHEMATICS

QUESTION ONE

(a) Using the table of standard integrals, you should then obtain:

$$\begin{aligned}\int_0^{\frac{\pi}{8}} \sec 2x \tan 2x \, dx &= \frac{1}{2}\left[\sec 2x\right]_0^{\frac{\pi}{8}} \\ &= \frac{1}{2}\left(\sec\frac{\pi}{4} - \sec 0\right) \\ &= \frac{1}{2}\left(\sqrt{2}-1\right).\end{aligned}$$

(b) $x = t^2+1 \Rightarrow \dfrac{dx}{dt} = 2t$

$\Rightarrow dx = 2t\,dt$

$\therefore\ t^2 = x-1$

$\therefore\ t = \sqrt{x-1}$ $\left(\text{do not need } -\sqrt{x-1}\right)$

$x = 2,\ t = 1$ and $x = 10,\ t = 3.$

$$\begin{aligned}\int_2^{10} \frac{x}{\sqrt{x-1}}\,dx &= \int_1^3 \frac{(t^2+1).2t}{t}\,dt \\ &= \int_1^3 (2t^2+2)\,dt \\ &= \left[\frac{2}{3}t^3+2t\right]_1^3 \\ &= \left(\frac{2}{3}\times 27+6\right)-\left(\frac{2}{3}+2\right) \\ &= 21\frac{1}{3}.\end{aligned}$$

(c) $P = \left(\dfrac{-2+30}{3+1}, \dfrac{3+33}{3+1}\right).$

$\therefore\ P = (7, 9).$

(d) $y = x \Rightarrow \dfrac{dy}{dx} = 1 = m_1$ (say).

$y = x^3 \Rightarrow \dfrac{dy}{dx} = 3x^2.$

At $x = 1$, $\dfrac{dy}{dx} = 3 = m_2$ (say).

Let the acute angle between the curves be α.

$$\therefore\ \tan\alpha = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

$$= \left|\frac{1-3}{1+1\times 3}\right|$$

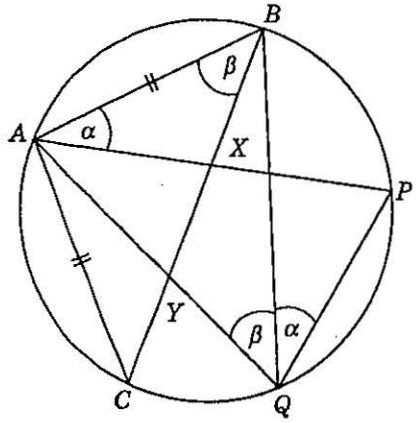
$\therefore\ \tan\alpha = \dfrac{1}{2}$

$\therefore\ \alpha = 26°\,34'$ (to nearest minute).

QUESTION TWO

(a) $$\begin{aligned}\text{LHS} &= \frac{2\tan A}{1+\tan^2 A} \\ &= \frac{2\tan A}{\sec^2 A} \\ &= 2\tan A\times\cos^2 A \\ &= \frac{2\sin A}{\cos A}\times\frac{\cos^2 A}{1} \\ &= 2\sin A\cos A \\ &= \sin 2A \\ &= \text{RHS}.\end{aligned}$$

(b)



(i) $\angle AXC = \alpha+\beta$ because $\angle AXC$ is the exterior angle of ΔAXB and is therefore equal to the sum of the two interior opposite angles, that is, $\alpha+\beta$.

(ii) $\angle BQP = \angle BAP$ (angles standing on same arc BP — or angles in the same segment are equal)

$\therefore\ \angle BQP = \alpha$

(iii) ΔABC is isosceles $(AC = AB)$

$\therefore\ \angle BCA = \angle ABC$ (angles opposite equal sides in isosceles Δ)

that is, $\angle BCA = \beta$.

Now $\angle BQA = \angle BCA$ (angles standing on same arc AB in same segment are equal)

$\therefore\ \angle BQA = \beta$.

(iv) $\angle AXY = \alpha + \beta$ (ext. $\angle$ of ΔABX)

$\angle AQP = \angle AQB + \angle PQB$

that is $\angle YQP = \beta + \alpha$ (from (ii); (iii))

$\therefore\ \angle AXY = \angle YQP$,

that is, exterior $\angle AXY$ of quadrilateral $PQYX$ is equal to opposite interior $\angle YQP$, that is, $PQYX$ is a cyclic quadrilateral.

OR from above:

$\angle PXY = 180° - \angle AXY$

$= 180° - (\alpha + \beta)$

$\therefore\ \angle PXY = \angle PQY = 180°$.

So $PQYX$ is a cyclic quadrilateral because opposite angles are supplementary.

(c) (i) *EITHER* test both $x = \frac{1}{3}$ *AND* $x = 2$ to show both satisfy $7 - 3x = \frac{2}{x}$:

that is, $x = \frac{1}{3}$,

LHS $= 7 - 1 = 6$

RHS $= 2 + \frac{1}{3} = 6$

$\therefore$ LHS = RHS for $x = \frac{1}{3}$.

$x = 2$,

LHS $= 7 - 6 = 1$

RHS $= \frac{2}{2} = 1$

$\therefore$ LHS = RHS for $x = 2$.

OR

$7 - 3x = \frac{2}{x}, \quad (x \neq 0)$

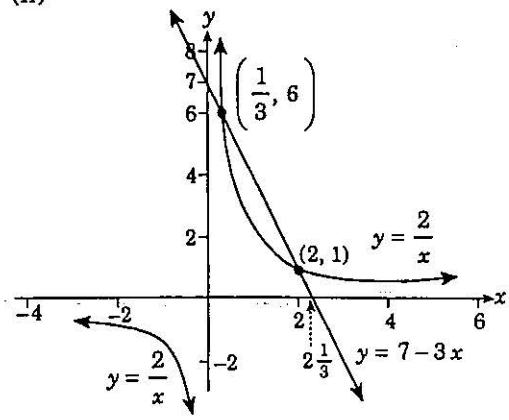
$7x - 3x^2 = 2$

$3x^2 - 7x + 2 = 0$

$(3x - 1)(x - 2) = 0$

$\therefore\ x = \frac{1}{3}$ or $x = 2$ as required.

(ii)



(iii) Using (ii): $7 - 3x < \frac{2}{x}$ means

'for what values of x is the graph of $y = 7 - 3x$ *below* the graph of $y = \frac{2}{x}$,

(noting that $y = \frac{2}{x}$ is undefined at $x = 0$).

That is, $0 < x < \frac{1}{3}$ or $x > 2$.

OR otherwise:

algebraically, $7 - 3x < \frac{2}{x}$.

$(\times x^2,\ x \neq 0)$:

$\therefore\ x^2(7 - 3x) < 2x$

$x^2(7 - 3x) - 2x < 0$

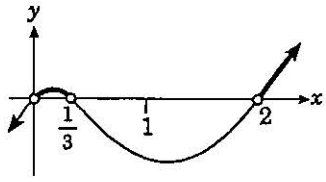
$x\left[x(7 - 3x) - 2\right] < 0$

$x(7x - 3x^2 - 2) < 0$

$x(3x^2 - 7x + 2) > 0$

$\therefore\ x(3x - 1)(x - 2) > 0$.

(*Note* There are other recognized methods for solving this type of inequality.)



that is, $0 < x < \frac{1}{3}$ or $x > 2$.

QUESTION THREE

(a) Need to consider binomial expansion of $(x + y)^{10}$, where x = probability of passing quality check, $\therefore\ x = 0{\cdot}75$;
and y = probability of failing quality check, $\therefore\ y = 0{\cdot}25$.

$\binom{10}{8} y^2 x^8 + \binom{10}{9} y^1 x^9 + \binom{10}{10} y^0 x^{10}$

$$= \binom{10}{8}(0{\cdot}25)^2(0{\cdot}75)^8 + \binom{10}{9}(0{\cdot}25)^1(0{\cdot}75)^9 + \binom{10}{10}(0{\cdot}75)^{10}$$

where $\binom{10}{8} = 45$, $\binom{10}{9} = 10$, and $\binom{10}{10} = 1$.

(b) Noting that $\frac{d}{dx}(\cos^3 x) = 3\cos^2 x \cdot -\sin x$

$$= -3\cos^2 x \sin x,$$

$$\therefore \int_0^{\frac{\pi}{3}} 3\sin x \cos^2 x \, dx = -\left[\cos^3 x\right]_0^{\frac{\pi}{3}}$$

$$= -\left(\left(\frac{1}{2}\right)^3 - (1)^3\right)$$

$$= \frac{7}{8}.$$

OR, Put $u = \cos x$.

$$\therefore \frac{du}{dx} = -\sin x \quad (\text{or } du = -\sin x \, dx)$$

$x = 0$, $u = \cos 0 = 1$

$x = \frac{\pi}{3}$, $u = \cos\frac{\pi}{3} = \frac{1}{2}$

$$\therefore \int_0^{\frac{\pi}{3}} 3\sin x \cos^2 x \, dx = -3\int_1^{\frac{1}{2}} u^2 \, du$$

$$= 3\int_{\frac{1}{2}}^1 u^2 \, du$$

$$= \left[u^3\right]_{\frac{1}{2}}^1$$

$$= 1^3 - \left(\frac{1}{2}\right)^3$$

$$= \frac{7}{8}.$$

(c) When $n = 1$, $n^3 + 2n = 1 + 2 = 3$, which is divisible by 3.

Assume $k^3 + 2k$ is divisible by 3, where k is a positive integer, that is $k^3 + 2k = 3M$; M is some positive integer.

Hence prove that $(k+1)^3 + 2(k+1)$ is divisible by 3.

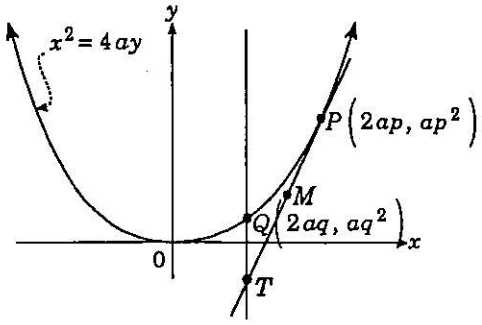
Now, $(k+1)^3 + 2(k+1)$

$$= k^3 + 3k^2 + 3k + 1 + 2k + 2$$
$$= (k^3 + 2k) + 3k^2 + 3k + 3$$
$$= 3M + 3(k^2 + k + 1)$$
$$= 3(M + k^2 + k + 1)$$

which is divisible by 3, that is true for $n = k + 1$.

$\therefore$ By the principle of mathematical induction, $n^3 + 2n$ is divisible by 3 for $n = 1, 2, \ldots$.

(d)



(i) $y = \frac{1}{4a}x^2$

$$\frac{dy}{dx} = \frac{1}{2a}x.$$

At P, $\frac{dy}{dx} = \frac{1}{2a} \times 2ap$

$$= p.$$

Eqn. of tangent is $y - ap^2 = p(x - 2ap)$

$$y - ap^2 = px - 2ap^2$$

$$\therefore \quad y = px - ap^2.$$

(ii) Q has coordinates $\left(2aq, aq^2\right)$.

Line through Q parallel to y axis has equation $x = 2aq$,

$$\therefore \quad y = px - ap^2 \qquad \text{—①}$$
$$x = 2aq. \qquad \text{—②}$$

② into ①:

$$y = 2apq - ap^2,$$

$\therefore$ T has coords $\left(2aq, 2apq - ap^2\right)$.

(iii) Coords. of M

$$= \left(\frac{2ap + 2aq}{2}, \frac{2apq - ap^2 + ap^2}{2}\right)$$

$$= \left(a(p+q), apq\right).$$

(iv) At M, $x = a(p+q)$ and $y = apq$.
Since $pq = -1$, we have $y = -a$,
that is, M lies on $y = -a$, the directrix of $x^2 = 4ay$.

QUESTION FOUR

(a) (i) $\frac{P(x)}{(x+1)(x-4)} = Qx + \frac{R(x)}{(x+1)(x-4)}$ —①

where the degree of $R(x)$ *must* be smaller than the degree of $(x+1)(x-4)$; that is, the degree of $R(x)$ must be smaller than 2;
that is, $R(x)$ must be of the form $ax + b$.

(ii) From ① in (i),

$$P(x) = (x+1)(x-4)\cdot Q(x) + R(x)$$

$\therefore \quad P(4) = 5\times 0\times Q(4) + R(4)$

that is, $P(4) = R(4)$, and $P(4) = -5$,

$\therefore \quad R(4) = -5.$

(iii) From (i), $R(x) = ax+b$.

$$R(4) = -5 \Rightarrow +4a+b = -5 \quad -①$$

$$R(-1) = 5 \Rightarrow -a+b = 5 \quad -②$$

①−②: $\quad 5a = -10$

$$a = -2$$

and thus $\quad b = 3,$

that is, $\quad R(x) = -2x+3.$

(b) Let $f(x) = x-3+e^{2x}$

$$f(1) = 1-3+e^2$$
$$= e^2-2$$
$$f'(x) = 1+2e^{2x}$$
$$f'(1) = 1+2e^2.$$

New approximation of x is given by:

$$x = 1-\frac{f(1)}{f'(1)}$$

$\therefore \quad x = 1-\dfrac{e^2-2}{1+2e^2}$

$\approx 0{\cdot}7$ (to 1 decimal place).

(c) (i) $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = 3x^2-12x$

$\therefore \quad \dfrac{1}{2}v^2 = x^3-6x^2+A$, A a constant.

$x = 0,\ v = 4\sqrt{2}$

$\therefore \dfrac{1}{2}\times 16\times 2 = A$

$$A = 16.$$

$\therefore \quad \dfrac{1}{2}v^2 = x^3-6x^2+16$

$\therefore \quad v^2 = 2(x^3-6x^2+16).$

(ii) When $x = 2$, $v^2 = 2(2^3-6\times 2^2+16 = 0$

$\therefore \quad v = 0,$

that is, velocity is zero at $x = 2$.

When $x = 2$, $\dfrac{d^2x}{dt^2} = -12$, that is,

acceleration is -12 m s^{-2} when $x = 2$.

(iii) At $x = 2$, velocity is 0, and acceleration is towards 0,

$\therefore$ particle P moves towards 0 from $x = 2$.

(iv) $\dfrac{d^2x}{dt^2} = 0 \Rightarrow x = 0$ or 4.

After it moves from $x = 2$, the particle picks up speed and moves ← until it gets to the origin, when it will still continue ← but will slow down until it stops, then it will move off again in → direction.

QUESTION FIVE

(a) *EITHER* use t-substitution:

Let $t = \tan\dfrac{\theta}{2}$ so that

$$\sin\theta = \frac{2t}{1+t^2} \text{ and } \cos\theta = \frac{1-t^2}{1+t^2}.$$

(Note that $0 \le \theta \le 2\pi$ means $0 \le \dfrac{\theta}{2} \le \pi$.)

$\therefore \quad \dfrac{2\sqrt{3}t}{1+t^2} - \dfrac{1-t^2}{1+t^2} = 1.$

$\times (1+t^2)$:

$$2\sqrt{3}t - 1 + t^2 = 1 + t^2$$

$\therefore \quad 2\sqrt{3}t = 2$

$$t = \frac{1}{\sqrt{3}},$$

that is $\quad \tan\dfrac{\theta}{2} = \dfrac{1}{\sqrt{3}}$ and $0 \le \dfrac{\theta}{2} \le \pi.$

$\therefore \quad \dfrac{\theta}{2} = \dfrac{\pi}{6}$

$$\theta = \frac{\pi}{3}.$$

Note that the 't-substitution' method fails when $\dfrac{\theta}{2} = \dfrac{\pi}{2}$, that is, when $\theta = \pi$, as $\tan\dfrac{\pi}{2}$ is undefined. So need to check $\theta = \pi$ separately.

LHS $= \sqrt{3}\sin\pi - \cos\pi = 0+1 = 1 =$ RHS,

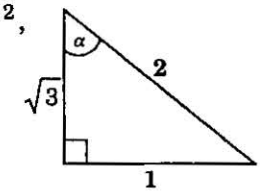
$\therefore \quad \theta = \pi$ is also a solution.

OR, Use subsidiary angle method:

$$\left(\sqrt{3}\right)^2 + 1^2 = 4 = 2^2,$$

$\therefore \sqrt{3}\sin\theta - \cos\theta = 1$ can be written as

$$\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta = \frac{1}{2},$$



that is, $\sin\theta\cos\alpha - \cos\theta\sin\alpha = \dfrac{1}{2}$ $\quad -①$

$$\sin\alpha = \frac{1}{2} \quad -②$$

$$\cos\alpha = \frac{\sqrt{3}}{2} \quad -③$$

so $\quad \tan\alpha = \dfrac{1}{\sqrt{3}}.$ $\quad -④$

$\therefore$ ① $\Rightarrow$ $\sin(\theta-\alpha)=\frac{1}{2}$,

and from ④: $\alpha=\frac{\pi}{6}$,

$\therefore$ $\sin\left(\theta-\frac{\pi}{6}\right)=\frac{1}{2}$,

$\therefore$ $\theta-\frac{\pi}{6}=\frac{\pi}{6}$ or $\frac{5\pi}{6}$,

$\therefore$ $\theta=\frac{\pi}{3}$ or π.

(b) $V=\pi\int_0^h x^2\,dy$

$=32\pi\int_0^h y\,dy$

$=16\pi\left[y^2\right]_0^h$

$\therefore$ $V=16\pi h^2$.

(c) (i) The rate of evaporation of the water is proportional to the surface area of the water—that is, at any time t (hours) the volume of water in the dam is decreasing at a rate proportional to the area of the surface of the water.

(ii) Note that if $y=h$, then $x^2=32h$ so S, which has a circular area with radius x, is equal to πx^2, that is, $S=32\pi h$.

Now, $\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$

$\therefore$ $-kS=32\pi h\times\frac{dh}{dt}$

where $\frac{dV}{dh}=32\pi h$,

that is, $-32\pi hk=32\pi h\times\frac{dh}{dt}$

$\therefore$ $\frac{dh}{dt}=-k$.

(iii) If $V=64\pi$,

then $16\pi h^2=64\pi$

$h^2=4$

$h=2$ here (since $h\neq -2$),

that is, when $t=0$, $h=2$.

$\frac{dh}{dt}=-k \Rightarrow h=-kt+c$, c a constant,

$\therefore$ $c=2$,

$\therefore$ $h=-kt+2$.

When dam is empty, $h=0$,

$\therefore$ $-kt+2=0$, and $k=0{\cdot}001$,

$\therefore$ $t=\frac{2}{0{\cdot}001}$.

That is, it will take 2000 hours for the dam to empty.

QUESTION SIX

(a) (i) $f(0)=0$, that is, y intercept is 0.

To find x intercepts, consider $3x-x^3=0$:

$\therefore$ $x(3-x^2)=0$

$x\left(\sqrt{3}-x\right)\left(\sqrt{3}+x\right)=0$.

$\therefore$ x intercepts are at 0, $\sqrt{3}$ and $-\sqrt{3}$.

$f'(x)=3-3x^2$

$=0$ for stationary points,

$\therefore$ $x^2=1$

$\therefore$ $x=1$ or -1,

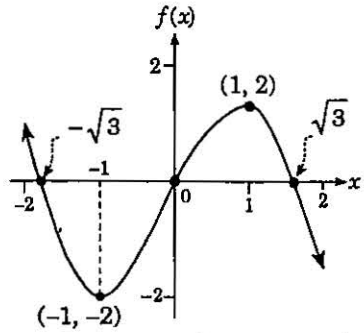
that is, (1, 2) and (−1, −2).

$f''(x)=-6x$

$f''(1)<0 \Rightarrow$ maximum turning point at (1, 2).

$f''(-1)>0 \Rightarrow$ minimum turning point at (−1, −2).

Note: $f(x)=3x-x^3$ is an ODD function.



(ii) $f(x)$ is a one-to-one function in the domain $-1\le x\le 1$. This means that $f(x)$ will have an inverse function $f^{-1}(x)$ in the domain $-1\le x\le 1$.

(iii) The domain of $f^{-1}(x)$ is $\{x:\ -2\le x\le 2\}$.

(iv) $f'(0)=3$

Now if $m_1=\frac{d}{dx}f(x)$ and

$m_2=\frac{d}{dx}\left[f^{-1}(x)\right]$,

then $m_1m_2=1$.

So if $m_1=3$, then $m_2=\frac{1}{3}$.

That is, the gradient of the inverse function at $x=0$ is $\frac{1}{3}$.

(b) (i) Total number of bins

$= 1+2+3+\cdots+r+\cdots+n,$

that is, sum of an AP where

$a = 1,\ d = 1,\ \ell = n,\ n = n.$

$$S_n = \frac{n}{2}\left[a+\ell\right]$$
$$= \frac{n}{2}(1+n).$$

$\therefore$ Total number of bins $= \dfrac{n(n+1)}{2}$, as required.

(ii) Total number of bottles

$$= 1(c+1)+2(c+2)+\cdots+r(c+r)+\cdots+n(c+n)$$
$$= (c+2c+\cdots+rc+\cdots+nc) + (1^2+2^2+\cdots+r^2+\cdots+n^2)$$
$$= c(1+2+\cdots+r+\cdots+n) + (1^2+2^2+\cdots+r^2+\cdots+n^2)$$
$$= c\frac{n(n+1)}{2}+\frac{1}{6}n(n+1)(2n+1), \text{ from part (i)}$$
$$= \frac{1}{6}n(n+1)\left\{3c+2n+1\right\}.$$

(iii) Average number of bottles per bin

$$= \frac{\text{total number of bottles}}{\text{total number of bins}}.$$

Given $c = 5$,

$$\frac{\dfrac{5n(n+1)}{2}+\dfrac{n(n+1)(2n+1)}{6}}{\dfrac{n(n+1)}{2}} = 10$$

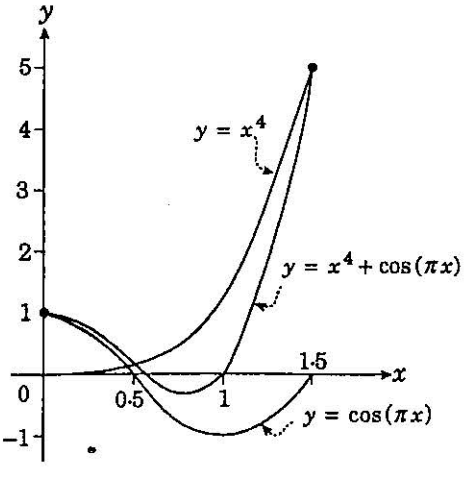
$$\therefore \quad \frac{15n(n+1)+n(n+1)(2n+1)}{3n(n+1)} = 10$$

$$\therefore \quad \frac{15+2n+1}{3} = 10$$
$$2n+16 = 30$$
$$n = 7.$$

$\therefore$ 7 rows in the storage rack.

QUESTION SEVEN

(a) (i) and (ii)



(iii) The graph of $y = x^4+\cos(\pi x)$ crosses the x axis in two places, for $x \geq 0$, that is, $x^4+\cos(\pi x) = 0$ has 2 positive real roots.

(b) (i) $\binom{5}{3} = 10$ sets of 3 points.

(ii) $\binom{9}{3}-\binom{5}{3} = 84-10$
$= 74.$

that is, 74 different triangles.

OR

No. of triangles with base on the line l
$= 10\times 4 = 40.$

No. of triangles with 1 vertex on l
$= 5\times {}^4C_2 = 5\times\dfrac{4\cdot 3}{1\cdot 2} = 5\times 6 = 30.$

No. of triangles with no vertex on l
$= {}^4C_3 = 4.$

Total no. of triangles = 74.

(c)

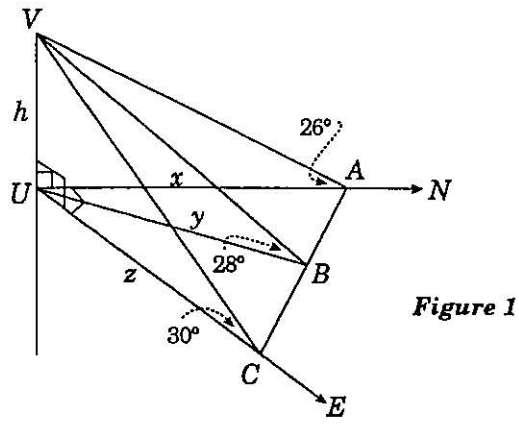


Figure 1

In Figure 1,

Let $UV = h$, $UA = x$, $UB = y$ and $UC = z$.

$\angle VUA = \angle VUB = \angle VUC = \angle AUC = 90°$,

$\therefore$ $\angle UVA = 64°$, $\angle UVB = 62°$, $\angle UVC = 60°$,

$\therefore$ In ΔUAV, $x = h\tan 64°$.

In ΔUBV, $y = h\tan 62°$.

In ΔUCV, $z = h\tan 60°$.

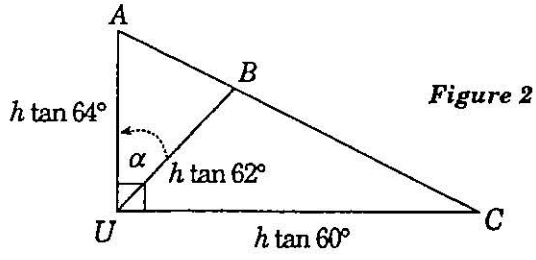


Figure 2

We need to find α.

In Figure 2,

in ΔAUC, $\tan \angle UAC = \dfrac{\tan 60°}{\tan 64°}$,

$\therefore$ $\angle UAC = 40° 11'$ (nearest minute);

$\therefore$ in ΔAUB,

$$\frac{h \tan 62°}{\sin 40° 11'} = \frac{h \tan 64°}{\sin \angle ABU}$$

$$\therefore \quad \sin \angle ABU = \frac{\tan 64° \sin 40° 11'}{\tan 62°}$$

$\therefore$ $\angle ABU = 44° 43'$ or $135° 17'$ (nearest minute),

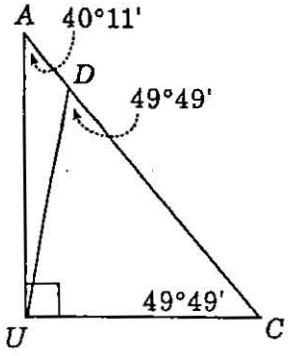
$\therefore$ $\alpha = 95° 6'$ or $4° 32'$.

$\alpha \neq 95° 7'$ as B is on the line-of-sight from A to C,

$\therefore$ $\alpha = 4° 32'$,

that is, B is 004° 32' from Uluru.

Note A first reaction to this answer may be that the angle is surprisingly small. To explain this, look at the diagram below, which shows the view from above.



Let D be the point on AC such that $UD = UC$. Then $\angle UDC = \angle UCD = 49° 49'$ (since $\angle UAC = 40° 11'$). Hence $\angle AUD$ is approximately 9°, and since B must lie between A and D, $\angle AUB$ must be less than 9°.