1992 HIGHER SCHOOL CERTIFICATE
SOLUTIONS
3/4 UNIT MATHEMATICS

QUESTION ONE

(a) $x^2 - x - 2 > 0$

$(x-2)(x+1) > 0$

$\therefore \quad x > 2, \quad x < -1$



(b) $\dfrac{d}{dx}\left(1+x^2\right)^{-\frac{1}{2}}$

$$= -\frac{1}{2}\left(1+x^2\right)^{-\frac{3}{2}} . 2x$$

$$= \frac{-x}{\sqrt{\left(1+x^2\right)^3}}$$

(c) $\displaystyle\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\, dx$

$$= \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_1^{\sqrt{3}}$$

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{3} - \frac{\pi}{6}$$

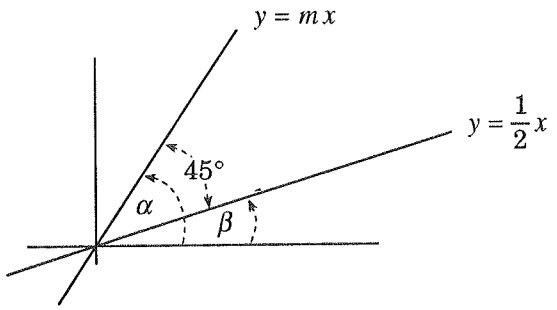
$$= \frac{\pi}{6}$$

(d) Need to select the appropriate term from the expansion of $(0{\cdot}2 + 0{\cdot}8)^{31}$.

p(10 rainy days exactly)

$$= \binom{31}{10}(0{\cdot}2)^{10}(0{\cdot}8)^{21}$$

(e)



$$\tan\beta = \frac{1}{2}$$

$$m = \tan\alpha = \tan\left(\beta + 45°\right)$$

$$= \frac{\tan\beta + \tan 45°}{1 - \tan\beta . \tan 45°}$$

$$= \frac{1+\tan\beta}{1-\tan\beta}$$

$$= \frac{1+\frac{1}{2}}{1-\frac{1}{2}} = \frac{1\frac{1}{2}}{\frac{1}{2}} = 3.$$

QUESTION TWO

(a) $2\sin^2\theta = \sin 2\theta \qquad$ for $0 \le \theta \le 2\pi$.

$$2\sin^2\theta - 2\sin\theta\cos\theta = 0$$

$$2\sin\theta\left(\sin\theta - \cos\theta\right) = 0$$

$\therefore \qquad \sin\theta = 0 \Rightarrow \theta = 0,\ \theta,\ 2\pi$

and/or $\qquad \sin\theta = \cos\theta \Rightarrow \dfrac{\sin\theta}{\cos\theta} = 1 \quad (\cos\theta \neq 0)$

$\therefore \qquad \tan\theta = 1$

$$\theta = \frac{\pi}{4},\ \frac{5\pi}{4}$$

$\therefore \qquad \theta = 0,\ \dfrac{\pi}{4},\ \pi,\ \dfrac{5\pi}{4},\ 2\pi.$

(b) $x = 3\cos \pi t$

$\dot{x} = -3\pi \sin \pi t$

$\ddot{x} = -3\pi^2 \cos \pi t$

$= -\pi^2 (3\cos \pi t)$

$\ddot{x} = -\pi^2 x$ which is of the form $\ddot{x} = -n^2 x$, and so is S.H.M. (as stated in the question) about 0.

(i) $T = \frac{2\pi}{n} = \frac{2\pi}{\pi} = 2$

Period is 2 secs.

(ii) When $x = 0$ (equilibrium position)

$3\cos \pi t = 0$

$\cos \pi t = 0.$

$\therefore \quad \pi t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots, \frac{(2n-1)\pi}{2}$

n integer.

$t = \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, \ldots, \frac{(2n-1)}{2}$

For these t values,

$\dot{x} = -3\pi \sin \frac{(2n-1)\pi}{2}$

$= \pm 3\pi.$

$\therefore$ Speed of particle is 3π m/sec.

(iii) From (b) above, $\ddot{x} = -\pi^2 x$

$\therefore \quad \ddot{x} \approx x$

i.e. acceleration is proportional to the displacement from the equilibrium position.

(c) Let $f(x) = x - 2\sin x$

$\therefore \quad f'(x) = 1 - 2\cos x.$

A better approximation to $x = 1{\cdot}7$ is $1{\cdot}7 - \frac{f(1{\cdot}7)}{f'(1{\cdot}7)}$

$= 1{\cdot}7 - \left(\frac{1{\cdot}7 - 2\sin 1{\cdot}7}{1 - 2\cos 1{\cdot}7}\right)$

$\approx 1{\cdot}7 - \left(\frac{1{\cdot}7 - 2 \times 0{\cdot}99166..}{1 - 2 \times -0{\cdot}1288..}\right)$

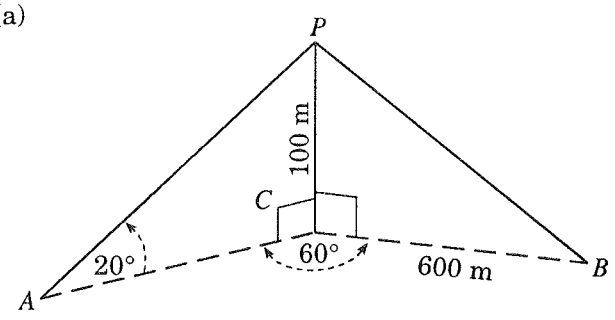
$= 1{\cdot}7 - \frac{-0{\cdot}2833..}{1{\cdot}2576..}$

$\approx 1{\cdot}7 + 0{\cdot}23$

$= 1{\cdot}93$

QUESTION THREE

(a)



In ΔPAC: $\frac{100}{AC} = \tan 20°$

$\therefore \quad AC = \frac{100}{\tan 20°}$

$\approx 274{\cdot}7$

In ΔABC:

$AB^2 = AC^2 + BC^2 - 2.AC.BC\cos 60°$

$= 274{\cdot}7^2 + 600^2 - 2 \times 274{\cdot}7 \times 600\cos 60°$

$AB^2 \approx 270\,640$

$\therefore \quad AB \approx 520$

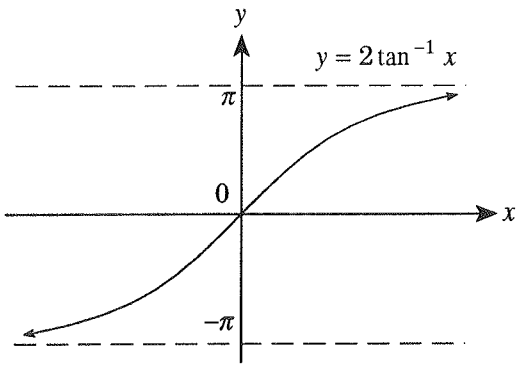
$\therefore$ Distance between yachts is approximately 520 m.

(b) $f(x) = 2\tan^{-1} x$

(i) $f(\sqrt{3}) = 2\tan^{-1}(\sqrt{3})$

$= 2.\frac{\pi}{3} = \frac{2\pi}{3}$

(ii)



Point of inflexion at (0, 0).

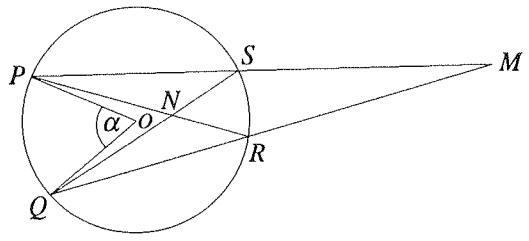
Asymptotes at $y = \pi$ and $y = -\pi$.

(iii) $y = 2\tan^{-1} x$

$\frac{dy}{dx} = 2.\frac{1}{1+x^2}$

$\frac{dy}{dx} = 2$

(c)



(i) $\angle PRQ = \frac{1}{2}.\angle POQ = \frac{1}{2}.\alpha$

(angle at the centre is twice the angle at the circumference standing on the same arc)

$\angle PRM = \pi - \angle PRQ$

($\angle QRM$ is a straight angle)

$\therefore \angle PRM = \left(\pi - \frac{\alpha}{2}\right)$.

(ii) Similarly $\angle QSM = \left(\pi - \frac{\alpha}{2}\right)$ (using $\angle PSQ$)

$\angle SNR + \angle NSM + \angle NRM + \angle RMS = 2\pi$

($MRNS$ quadrilateral)

$\therefore \quad \angle SNR + \angle RMS + 2\left(\pi - \frac{\alpha}{2}\right) = 2\pi$

i.e. $\angle SNR + \angle RMS = \alpha$.

But $\angle PNQ = \angle SNR$ (vertically opposite $\angle$s)

$\therefore \quad \angle PNQ + \angle PMQ = \alpha$.

QUESTION FOUR

(a) $\displaystyle\int_0^1 \frac{2x}{(2x+1)^2}\,dx$ Let $u = 2x+1 \Rightarrow x = \left(\frac{u-1}{2}\right)$

$du = 2dx \Rightarrow dx = \frac{du}{2}$

$$= \int_1^3 \frac{2\left(\frac{u-1}{2}\right)}{u^2}\cdot\frac{du}{2} \qquad \begin{cases} x = 0, & u = 1 \\ x = 1, & u = 3 \end{cases}$$

$$= \int_1^3 \frac{u-1}{2u^2}\,du$$

$$= \int_1^3 \left(\frac{u}{2u^2} - \frac{1}{2u^2}\right)du$$

$$= \int_1^3 \left(\frac{1}{2u} - \frac{1}{2u^2}\right)du$$

$$= \left[\frac{1}{2}\log u + \frac{1}{2u}\right]_1^3$$

$$= \left(\frac{1}{2}\log 3 + \frac{1}{6}\right) - \left(\frac{1}{2}\log 1 + \frac{1}{2}\right)$$

$$= \frac{1}{2}\log 3 + \frac{1}{6} - \frac{1}{2}$$

$$= \frac{1}{2}\log 3 - \frac{1}{3}$$

(b) $S_n = 1\times 2 + 2\times 3 + \ldots + (n-1)n$

$S_2 = 1\times 2 = 2$

For $S_n = \frac{1}{3}(n-1)n(n+1)$

$S_2 = \frac{1}{3}\times 1\times 2\times 3 = 2$

$\therefore$ The formula is correct for $n = 2$.

Assume that $1\times 2 + 2\times 3 + \ldots + (n-1)n$

$= \frac{1}{3}(n-1)n(n+1)$,

$\therefore \quad 1\times 2 + 2\times 3 + \ldots + (n-1)n + n(n+1)$

$$= \frac{1}{3}(n-1)n(n+1) + n(n+1)$$

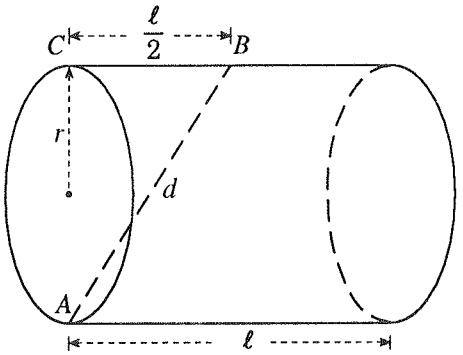
$$= \frac{1}{3}n(n+1)\{(n-1)+3\}$$

$$= \frac{1}{3}n(n+1)(n+2)$$

$$= \frac{1}{3}(\overline{n+1}-1)(\overline{n+1})(\overline{n+1}+1)$$ which is of the expected form, i.e. with n replaced by $n+1$, i.e. if the result is true for some integer, n, then it is true for the next integer, $(n+1)$. The result is true for $n = 2$ (proved) and so is true for $n = 3$ and this in turn proves it true for the next integer, $n = 4$, and so on forever.

i.e. $S_n = \frac{1}{3}(n-1)n(n+1)$ for $n \geq 2$.

(c)



(i) By Pythagoras from ΔABC,

$$d^2 = \left(\frac{\ell}{2}\right)^2 + (2r)^2$$

$$d^2 = \frac{\ell^2}{4} + 4r^2$$

$$\therefore \quad r^2 = \frac{1}{4}\left(d^2 - \frac{\ell^2}{4}\right)$$

Volume of cylinder, $V = \pi r^2 \ell$

$$= \pi\left[\frac{1}{4}\left(d^2 - \frac{\ell^2}{4}\right)\right]\ell$$

$$= \frac{\pi\ell}{4}\left(d^2 - \frac{\ell^2}{4}\right)$$

(ii) $V = \frac{\pi d^2}{4}\ell - \frac{\pi}{16}\ell^3$, d is constant

$$\frac{dV}{d\ell} = \frac{\pi d^2}{4} - \frac{3\pi}{16}\ell^2$$

$\frac{dV}{d\ell} = 0$ gives stationary points.

$$\frac{\pi d^2}{4} - \frac{3\pi}{16}\ell^2 = 0$$

$$\therefore \quad d^2 = \frac{3}{4}\ell^2$$

$$d = \frac{\sqrt{3}}{2}\ell.$$

$$\frac{d^2V}{d\ell^2} = \frac{-3\pi}{8}\ell < 0 \text{ when } d = \frac{\sqrt{3}}{2}\ell$$

$\therefore$ maximum when $d = \frac{\sqrt{3}}{2}\ell$

i.e. $\ell = \frac{2d}{\sqrt{3}}$.

QUESTION FIVE

(a)(i) A parabola is the locus of a point which moves so that its distance from a fixed point called the focus is equal to its distance from a fixed line, called the directrix.
P is on the parabola. S is the focus. D is on the directrix and $PD \perp$ directrix,
$\therefore SP = PD$.

(ii) At P: $x = 2at$; $y = at^2$;

$$\frac{dx}{dt} = 2a; \quad \frac{dy}{dt} = 2at.$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 2at \times \frac{1}{2a} = t$$

$\therefore$ Gradient of tangent at P is t.

(iii) $S(o, a)$; $D(2at, -a)$

$$m_{SD} = \frac{a - -a}{o - 2at} = \frac{2a}{-2at} = \frac{-1}{t}.$$

(iv) $m_{PQ} . m_{SD} = t . \frac{-1}{t} = -1.$

$\therefore PQ \perp SD$.

(v) In ΔSPD: $SP = PD$ from (i)
$PQ \perp SD$ from (iv)
$\therefore PQ$ is the perpendicular from the vertex of the isosceles triangle.

$$\left[\begin{array}{l} \Delta SPQ \equiv \Delta DPQ \text{ (RHS)} \\ SP = DP \\ PQ \text{ common} \\ \angle PQS = \angle PQD \end{array}\right]$$

$\therefore \quad \angle SPQ = \angle DPQ$

But $\angle RPT = \angle DPQ$ (vertically opposite angles)

$\therefore \quad \angle SPQ = \angle RPT$.

(b) $P = \frac{1000}{1 + ce^{-1000t}}$

(i) Eventually translates mathematically to $t \to \infty$.
As $t \to \infty$, $e^{-1000t} \to 0$.
i.e. eventually all 1000 chickens will be affected.

(ii) When $t = 0$, $P = 1$

$$1 = \frac{1000}{1 + c.1} \Rightarrow 1 + c = 1000 \Rightarrow c = 999$$

i.e. $P = \frac{1000}{1 + 999e^{-1000t}}$.

When $P = 500$: $500 = \frac{1000}{1 + 999e^{-1000t}}$

$$\therefore \quad 1 + 999e^{-1000t} = \frac{1000}{500}$$

$$999e^{-1000t} = 2 - 1$$

$$e^{-1000t} = \frac{1}{999}$$

$$e^{1000t} = 999$$

$$1000t = \log 999$$

$$t = 0{\cdot}006\,907 \text{ (years)}$$

$$= 0{\cdot}006\,907 \times 365 \text{ (days)}$$

$$\approx 2{\cdot}5 \text{ days.}$$

i.e. 500 chickens are affected after $2\frac{1}{2}$ days.

(iii) $P = 1000\left(1+ce^{-1000t}\right)^{-1}$

$$\frac{dP}{dt} = 1000.-1\left(1+ce^{-1000t}\right)^{-2}.(-1000c)e^{-1000t}$$

$$= \frac{10^6\, ce^{-1000t}}{\left(1+ce^{-1000t}\right)^2}$$

$$= \left\{\frac{1000}{\left(1+ce^{-1000t}\right)}\right\}^2 ce^{-1000t}$$

$$\left[\begin{array}{rl} & \left(1+ce^{-1000t}\right)^{-1} = \dfrac{P}{1000} \\ \therefore & 1+ce^{-1000t} = \dfrac{1000}{P} \\ & ce^{-1000t} = \dfrac{1000}{P} - 1 \end{array}\right]$$

$$= P^2\left(\frac{1000}{P} - 1\right)$$

$$= P(1000 - P).$$

QUESTION SIX

(a) $P(x) = x^n(2^m-1) + x^m(1-2^n) + (2^n-2^m)$

$P(1) = 1.(2^m-1) + 1.(1-2^n) + (2^n-2^m)$

$= 2^m - 2^n + 2^n - 2^m$

$= 0$

$P(2) = 2^n(2^m-1) + 2^m(1-2^n) + (2^n-2^m)$

$= 2^{m+n} - 2^n + 2^m - 2^{m+n} + 2^n - 2^m$

$= 0$

$\therefore$ $(x-1)$ and $(x-2)$ are factors and so $(x-1)(x-2)$ is a factor of $P(x)$.

(b)(i) The 5 selected players come from 4 teams of 10 (40 players) of whom 5 have a no. 6 and 5 others have a no. 8.

No. of possible selections of any 5 $= \binom{40}{5}$.

No. of selections of 3 'sixes' from 4 and 2 'eights' from 4 $= \binom{4}{3}\binom{4}{2}$.

$$\therefore \text{ prob.} = \frac{\binom{4}{3}\binom{4}{2}}{\binom{40}{5}} = \frac{4\times 6}{658\,008} = \frac{1}{27\,417}.$$

(ii) Firstly the four players from one team:

(A) There are 4 choices of team to choose 1 from:

$$\binom{4}{1}$$

(B) Having selected the team, need to select 4 of the 10 players:

$$\binom{10}{4}$$

(C) The fifth player comes from the remaining 30 players not in the chosen team:

$$\binom{30}{1}$$

[or choose team of 3 left: $\binom{3}{1}$ then 1 of 10 from that team:

$$= \binom{10}{1} \Rightarrow \binom{3}{1}\binom{10}{1} = 3\times 10 = 30]$$

No. ways $= \binom{4}{1}\binom{10}{4}\binom{30}{1}$

$= 4\times 210\times 30 = 25\,200.$

Five players from the same team:

(A) 4 choices of team: $\binom{4}{1}$

(B) Select 5 of 10 players: $\binom{10}{5}$

No. ways $= \binom{4}{1}\binom{10}{5}$

$= 4\times 252 = 1008.$

$\therefore$ p (at least 4 from same team)

$$= \frac{\binom{4}{1}\binom{10}{4}\binom{30}{1} + \binom{4}{1}\binom{10}{5}}{\binom{40}{5}}$$

/ continued over

$$= \frac{25200 + 1008}{658008}$$

$$= \frac{26208}{658008}$$

$$= \frac{28}{703}$$

$$\approx \frac{1}{25}.$$

(c) $1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n = (1+x)^n$

(i) Let $x = -1$:

$$1 - \binom{n}{1} + \binom{n}{2} - \ldots + \binom{n}{n}(-1)^n = 0.$$

(ii) Integrate w.r. to x:

$$\int 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n \; dx$$

$$= \int (1+x)^n \; dx.$$

$$x + \frac{1}{2}\binom{n}{1}x^2 + \frac{1}{3}\binom{n}{2}x^3 + \ldots + \frac{1}{(n+1)}\binom{n}{n}x^{n+1}$$

$$= \frac{(1+x)^{n+1}}{(n+1)} + c$$

which is true for all values of x.

When $x = 0$: $\quad 0 = \dfrac{1}{(n+1)} + c$

$$\therefore \quad c = \frac{-1}{(n+1)}$$

$$\therefore \quad x + \frac{1}{2}\binom{n}{1}x^2 + \ldots + \frac{1}{(n+1)}\binom{n}{n}x^{n+1}$$

$$= \frac{(1+x)^{n+1}}{(n+1)} - \frac{1}{(n+1)}$$

When $x = -1$:

$$-1 + \frac{1}{2}\binom{n}{1} - \ldots + \frac{1}{(n+1)}\binom{n}{n}(-1)^{n+1}$$

$$= 0 - \frac{1}{(n+1)}$$

$$\therefore \quad 1 - \frac{1}{2}\binom{n}{1} + \ldots + \frac{1}{(n+1)}\binom{n}{n}\frac{(-1)^{n+1}}{(-1)}$$

$$= \frac{1}{(n+1)}.$$

QUESTION SEVEN

(a)(i) $\quad f(\theta) = \cos\theta - \dfrac{1}{4\sqrt{3}\sin\theta}$

$$= \cos\theta - \frac{1}{4\sqrt{3}}(\sin\theta)^{-1}$$

$$f'(\theta) = -\sin\theta - \frac{1}{4\sqrt{3}}.-1(\sin\theta)^{-2}.\cos\theta$$

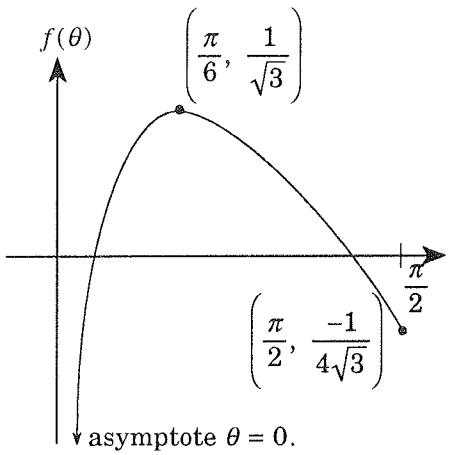
$$= -\sin\theta + \frac{\cos\theta}{4\sqrt{3}(\sin\theta)^2}$$

$$f'\left(\frac{\pi}{6}\right) = -\frac{1}{2} + \frac{\frac{\sqrt{3}}{2}}{4\sqrt{3}.\left(\frac{1}{2}\right)^2}$$

$$= -\frac{1}{2} + \frac{\frac{\sqrt{3}}{2}}{\sqrt{3}}$$

$$= -\frac{1}{2} + \frac{1}{2} = 0$$

(ii)



$$f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{1}{4\sqrt{3}.\frac{1}{2}}$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2\sqrt{3}}$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2\sqrt{3}}.\frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{6}$$

$$= \sqrt{3}\left(\frac{1}{2} - \frac{1}{6}\right)$$

$$= \sqrt{3}.\frac{1}{3} = \frac{1}{\sqrt{3}}$$

(b) (i) $x = Vt\cos\alpha \qquad y = -\frac{1}{2}gt^2 + Vt\sin\alpha$

Above x-axis: $y > 0$. At y-axis: $y = 0$.

$\therefore \quad t\left(V\sin\alpha - \frac{1}{2}gt\right) = 0$

$\therefore \quad t = \frac{2V\sin\alpha}{g}, \quad 0.$

Above x-axis for $\frac{2V\sin\alpha}{g}$ secs.

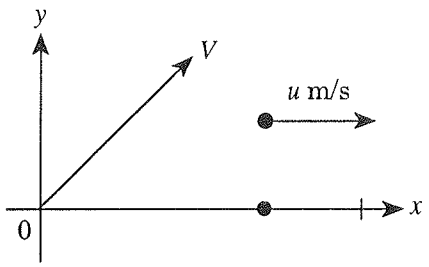
(ii) When $t = \frac{2V\sin\alpha}{g}$

$$x = V.\frac{2V\sin\alpha}{g}.\cos\alpha$$

$$= \frac{2V^2}{g}\sin\alpha\cos\alpha \text{ metres,}$$

which is the horizontal range.

(iii)



After t secs. the target has moved ut metres and the projectile has moved $Vt\cos\alpha$ metres. The target is d metres from 0 initially.

For collision: $\quad Vt\cos\alpha = ut + d$

$$\therefore \qquad u = \frac{Vt\cos\alpha - d}{t}$$

$$= V\cos\alpha - \frac{d}{t}$$

$$= V\cos\alpha - \frac{gd}{2V\sin\alpha}$$

$$\left(\because \quad t = \frac{2V\sin\alpha}{g}\right).$$

(iv) $gd = \frac{V^2}{2\sqrt{3}} \quad \therefore \quad u = V\cos\alpha - \frac{V^2}{2\sqrt{3}}.\frac{1}{2V\sin\alpha}$

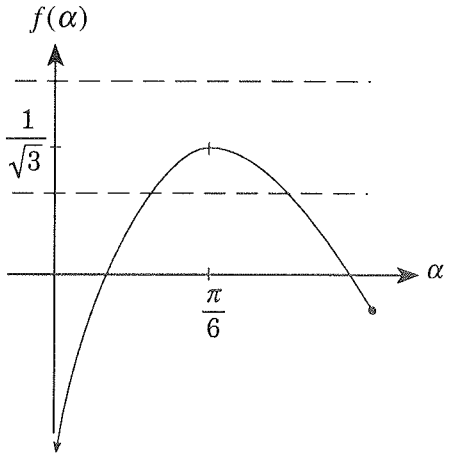
$$= V\cos\alpha - \frac{V}{4\sqrt{3}\sin\alpha}$$

$$= V\left(\cos\alpha - \frac{1}{4\sqrt{3}\sin\alpha}\right)$$

$$= V f(\alpha).$$

The maximum value of $f(\alpha)$ for $0 < \alpha < \frac{\pi}{2}$ is $\frac{1}{\sqrt{3}}$ for $\alpha = \frac{\pi}{6}$.

$\therefore \quad u < \frac{1}{\sqrt{3}}$ for all $\alpha \left(0 < \alpha < \frac{\pi}{2}\right)$.



So for $u > \frac{V}{\sqrt{3}}$ there exists no value of α.

i.e. there can be no collision of the target and projectile.

(v) For $u < \frac{V}{\sqrt{3}}$ (and for $u > 0$) there exist exactly two values of α (from graph for $0 < f(\alpha) < \frac{1}{\sqrt{3}} \Rightarrow$ two values of α).

Position of target

$$= ut + d = \frac{2uV\sin\alpha}{g} + d$$

which is of the form $k\sin\alpha + d$.

$\sin\alpha$ is a monotonically increasing function in $0 < \alpha < \frac{\pi}{2}$.

$\therefore$ Two values of $\alpha\left(0 < \alpha < \frac{\pi}{2}\right)$ implies two separate positions of the target.