1991 HIGHER SCHOOL CERTIFICATE SOLUTIONS

3/4 UNIT MATHEMATICS

QUESTION ONE

(a) (i) $\displaystyle\int_0^1 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_0^1 \frac{2x}{x^2+1}\,dx$

$\displaystyle = \frac{1}{2}\int_0^1 \frac{f'(x)}{f(x)}\,dx$

$\displaystyle = \frac{1}{2}\left[\ln\left(x^2+1\right)\right]_0^1$

$\displaystyle = \frac{1}{2}\ln 2$

(ii) $\displaystyle\int_{-1}^{1} (1+5x)^4\,dx.$

Let $u = 5x+1 \Rightarrow du = 5dx$

When $x = 1,\ u = 6$

When $x = -1,\ u = -4$

$\therefore\ \displaystyle\int_{-1}^{1} (1+5x)^4\,dx$

$\displaystyle = \int_{-4}^{6} \frac{u^4}{5}\,du$

$\displaystyle = \left[\frac{u^5}{25}\right]_{-4}^{6}$

$= 352$

(b) $P(x) = x^3 + ax + 12$

$P(-3) = 0 \Rightarrow -27 - 3a + 12 = 0$

$3a = -15$

$a = -5$

(c) As external k will be negative

$PB = \sqrt{81+144} = \sqrt{225} = 15,$

$PA = \sqrt{9+16} = 5$

$\dfrac{k}{1} = \dfrac{-15}{5} = -3$

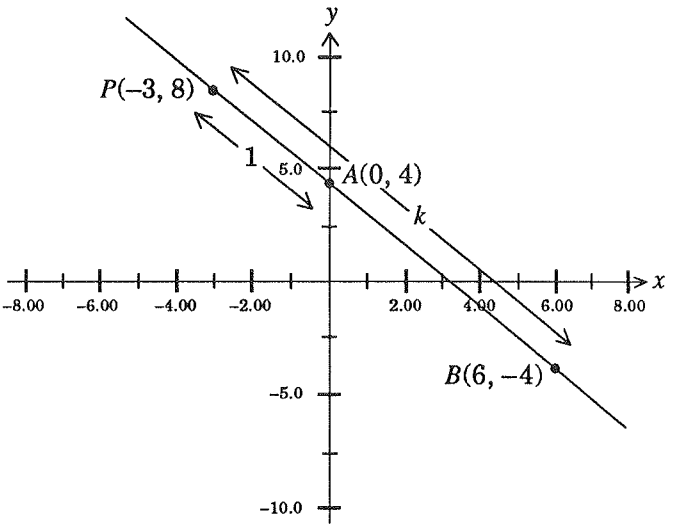
$k = -3$

or we can use the formula:

$\dfrac{kx_2 + 1x_1}{k+1} = -3$

$\dfrac{0+6}{k+1} = -3$

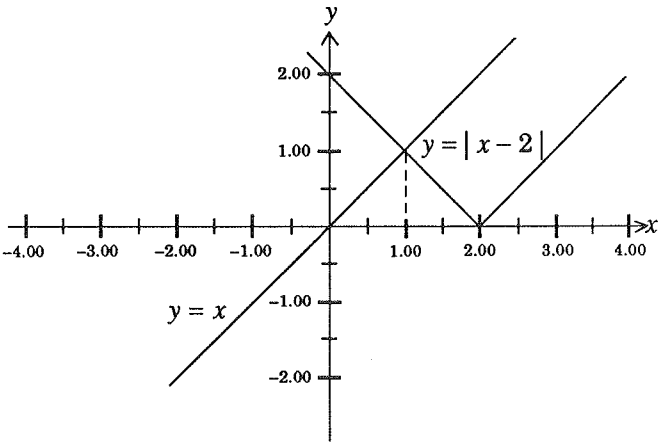
$\Rightarrow k = -3$



(d) (i) $y = |x-2|$

$y = x-2$ for $x \geq 2$

$y = 2-x$ for $x < 2$



(ii) From the graph, if $y = |x-2| < x$,

$x > 1$, or

$x - 2 < x$ when $x \geq 2$

$2 - x < x$ when $x < 2$

$\Rightarrow x > 1$

$\therefore\ x > 1$

QUESTION TWO

(a) (i) $y = e^{kx}$

$$\frac{dy}{dx} = ke^{kx}$$

$$\frac{d^2y}{dx^2} = k^2e^{kx}$$

(ii) If $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 0$

$$k^2e^{kx} + 7ke^{kx} + 12e^{kx} = 0$$

$$e^{kx}(k^2 + 7k + 12) = 0$$

$$(k+4)(k+3) = 0$$

$\therefore \quad k = -3, \ -4$

(b) (i) $\left(\frac{3}{4} + \frac{1}{4}\right)^5 P_{\text{All tall}} = {}^5C_5\left(\frac{3}{4}\right)^5\left(\frac{1}{4}\right)^0$

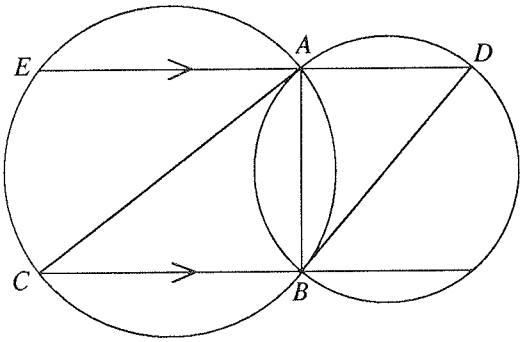
$$= \left(\frac{3}{4}\right)^5$$

(ii) $P_{\text{At least 3}} = P_{3 \text{ tall}} + P_{4 \text{ tall}} + P_{5 \text{ tall}}$

$$= {}^5C_3\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 + {}^5C_4\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^1 + \left(\frac{3}{4}\right)^5$$

$$= 34 \times \frac{3^3}{4^5}$$

(c) (i)



(ii) $\angle ABC = 90°$ (angle in semicircle)

$\therefore \quad \angle DAB = 90°$ (alternate angles and $AD \parallel CB$)

As $\angle$ in semicircle $= 90°$, BD is the diameter of circle ABD.

(iii) If $AC \parallel BD$, $ACBD$ is a parallelogram.

$\therefore \quad AC = BD$ (opposite sides of parm.)

AC and BD are equal diameters

$\therefore$ radii equal.

QUESTION THREE

(a) $x_2 = x_1 = \frac{f(x_1)}{f'(x_1)}$. Let $f(x) = x + \ln x$

$\therefore \quad x_2 = 0{\cdot}5 = \frac{-0{\cdot}193}{3} \qquad f(0{\cdot}5) = -0{\cdot}193\ldots$

$\cong 0{\cdot}56 \qquad f'(x) = 1 + \frac{1}{x}$

$f'(0{\cdot}5) = 3$

$\therefore \quad x_2 = 0{\cdot}56$ corr. to 2 dec. pl.

(b) $\int_0^{\frac{\pi}{2}} \sin^2 3x \ dx \qquad \cos 6x = 1 - 2\sin^2 3x$

$$\sin^2 3x = \frac{1 - \cos 6x}{2}$$

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} (1 - \cos 6x) \ dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 6x}{6}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

(c) $y = 10^x$

$\therefore \quad \ln y = \ln 10^x = x \ln 10$

$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \ln 10$

$$\frac{dy}{dx} = 10^x \ln 10$$

$\therefore \quad f'(1) = 10 \ln 10$

or use formula $a^x = e^{x \ln a}$, i.e. $10^x = e^{x \ln 10}$.

(d) (i) $V = \frac{4}{3}\pi r^3$ with $\frac{dV}{dt} = 200$

$$\frac{dV}{dr} = 4\pi r^2 \qquad \frac{dr}{dV} = \frac{1}{4\pi r^2}$$

$$\therefore \quad \frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

$$= \frac{1}{4\pi r^2} \times 200 = \frac{50}{\pi r^2}$$

(ii) $\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$ with $\frac{dS}{dr} = 8\pi r$

At $r = 50$, $\frac{dS}{dr} = 400\pi$

$$\frac{dS}{dt} = \frac{400\pi \times 50}{2500\pi}$$

$$= 8 \text{ mm}^2 \text{ s}^{-1}$$

QUESTION FOUR

(a) **Step 1** Test for $n = 1$.

L.H.S. is $2^0 = 1$, R.H.S. is $2^1 - 1 = 1$

$\therefore$ true for $n = 1$.

Step 2 Assume true for $n = k$,

i.e. $S_k = 2^k - 1$

Need to prove $S_{k+1} = 2^{k+1} - 1$

$$\begin{aligned} S_{k+1} &= 2^{k+1} + U_{k+1} \\ &= 2^k - 1 + 2^k \\ &= 2 \times 2^k - 1 \\ &= 2^{k+1} - 1 \end{aligned}$$

Step 3 We have shown that the result is true for $n = 1$ and that it is true for $n = k$ and $n = k + 1$.
$\therefore$ true for $n = 1 + 1 = 2$, $n = 2 + 1 = 3$, etc.
$\therefore$ have shown by induction the result is true for all k.

(b) $\dfrac{d^2x}{dt^2} = 2x - 3$

(i) $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = 2x - 3$

$$\frac{1}{2}v^2 = x^2 - 3x + c$$

When $t = 0$, $v = 0$, and $x = 4$

$\therefore \quad 16 - 12 + c = 0$

$$c = -4$$

$\therefore \quad v^2 = 2(x^2 - 3x - 4)$

(ii) If $x = 0$, $v^2 = -8$ not possible.
$\therefore$ Particle does not pass through the origin.

(iii) If $v = 10$, $100 = 2(x^2 - 3x - 4)$

$\therefore \quad x^2 - 3x - 4 - 50 = 0$

$$(x - 9)(x + 6) = 0$$

$\therefore \quad x = 9, \quad x \neq -6$ see (ii).

(c) (i) R, R, R, W, W, B $\dfrac{6!}{3!2!} = \dfrac{720}{12} = 60$

(ii) R, R, R, W, W $\dfrac{5!}{3!2!} = \dfrac{120}{12} = 10$

R, R, W, W, B $\dfrac{5!}{2!2!} = \dfrac{120}{4} = 30$

R, R, R, W, B $\dfrac{5!}{3!} = \dfrac{120}{6} = 20$

Total = 60

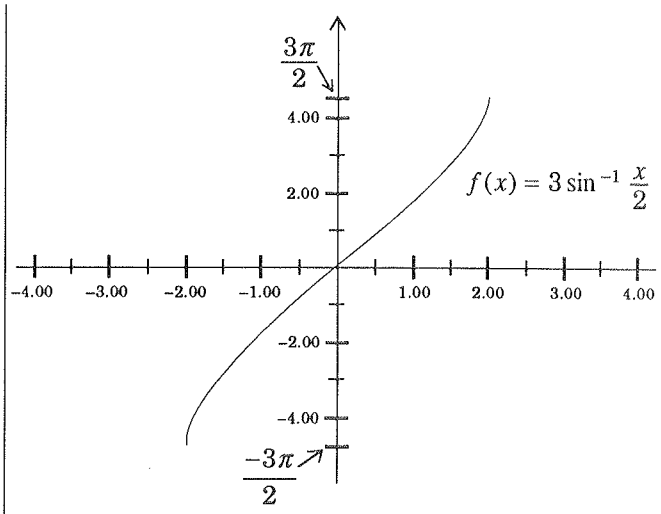
QUESTION FIVE

(a) $f(x) = 3\sin^{-1}\dfrac{x}{2}$

(i) $f(2) = 3\sin^{-1} 1$

$$= \frac{3\pi}{2}$$

(ii) See following diagram.



(iii) Domain: $|x| \leq 2$

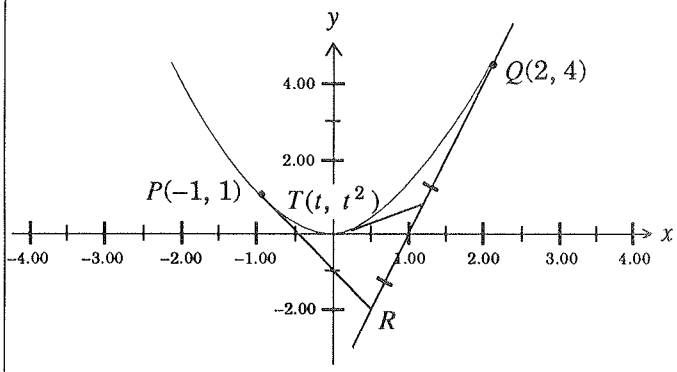
Range: $|f(x)| \leq \dfrac{3\pi}{2}$

(b) (i) Substitute $x = t$ into $y = t^2$

$\therefore$ required parabola is $y = x^2$

If $t = -1$ P is $(-1, 1)$.

If $t = 2$ Q is $(2, 4)$.



(ii) $f'(x) = 2x$, $f'(2) = 4$ and $f'(-1) = -2$

$\therefore$ tangents are $y - 1 = -2(x + 1)$,

$y + 2x = -1$ and $y - 4 = 4(x - 2)$,

$y - 4x = -4$.

Solving simultaneously:

$y + 2x = -1$ ...(1)

$y - 4x = -4$ ...(2)

Subtract $\Rightarrow 6x = 3$, $\quad x = \dfrac{1}{2}$

Subst. in (1): $y + 1 = -1$, $\quad y = -2$

$\therefore \quad R$ is $\left(\dfrac{1}{2}, -2\right)$.

(iii) Gradient of tangent is $\dfrac{dy}{dx}$

$$= \frac{dy}{dt} \times \frac{dt}{dx} = 2t$$

$\therefore$ tangent is $y - t^2 = 2t(x - t)$

Mid-point of QR is $\left(\frac{5}{4}, 1\right)$ which must satisfy tangent.

$\therefore \quad 1 - t^2 = 2t\left(\frac{5}{4} - t\right)$

$\Rightarrow 2t^2 - 5t + 2 = 0 \; (2t - 1)(t - 2) = 0$

Required solution is $t = \frac{1}{2}$.

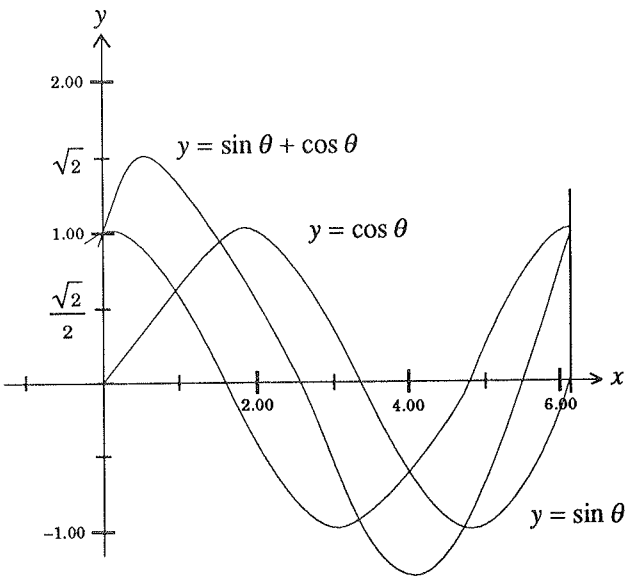
Slope of tangent at t is $2 \times \frac{1}{2} = 1$

$\therefore$ slope of $PQ = \frac{4 - 1}{3} = 1$

$\therefore$ $PQ \,||$ tangent at T.

QUESTION SIX

(a) (i)



(ii) $\sin x + \cos x = 1$ for $x = 0, \; \frac{\pi}{2}, \; 2\pi$

(iii) From the graph $y = \sin x + \cos x$

$|k| < \sqrt{2} \quad k \neq 1$, or

$\sin x + \cos x = \sqrt{2}\left(\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x\right)$

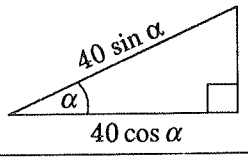
i.e. $y = \sqrt{2} \sin\left(\frac{\pi}{4} + x\right) = k$

$\therefore$ amplitude $= \sqrt{2}$

$\therefore \; |k| < \sqrt{2} \quad k \neq 1$

(b) Assume (0, 0) is at foot of building and up is positive.

(i)



$$\frac{d^2y}{dt^2} = -10$$

$$\frac{dy}{dt} = -10t + c$$

$$\frac{dy}{dt} = -10t + 40 \sin \alpha$$

$$y = -5t^2 + 40t \sin \alpha + e$$

If $y = 50$, when $t = 0$

$\therefore \; y = -5t^2 + 40t \sin \alpha + 50$

$$\frac{d^2x}{dt^2} = 0$$

$$\frac{dx}{dt} = d$$

$$\frac{dx}{dt} = 40 \cos \alpha$$

$$x = 40t \cos \alpha + f$$

When $t = 0, \quad x = 0$

$\therefore \; x = 40t \sin \alpha$

(ii) Substitute $t = \frac{x}{40 \cos \alpha}$ in y.

$\therefore \; y = \frac{-5x^2}{1600 \cos^2 \alpha} + \frac{40x \sin \alpha}{40 \cos \alpha} + 50$

$\therefore \; y = \frac{-x^2 \sec^2 \alpha}{320} + x \tan \alpha + 50$

At (200, 0)

$0 = \frac{-40\,000(\tan^2 \alpha + 1)}{320} + 200 \tan \alpha + 50$

$\Rightarrow 5 \tan^2 \alpha - 8 \tan \alpha + 3 = 0$

$\therefore \; (5 \tan \alpha = 3)(\tan \alpha - 1) = 0$

$\therefore \; \tan \alpha = 1, \quad \tan \alpha = \frac{3}{5}$

$\Rightarrow \alpha = 45°, \; 31°$

QUESTION SEVEN

(a) $f(x) = \frac{x}{x^2 - 1} = \frac{x}{(x - 1)(x + 1)}$

(i) Undefined for $x = \pm 1$.

(ii) $f(-x) = \frac{-x}{(-x)^2 - 1} = \frac{-x}{x^2 - 1} = -f(x)$

As $f(x) = -f(x)$, the function is odd.

(iii) $f'(x) = \frac{(x^2 - 1)1 - x \times 2x}{(x^2 - 1)^2}$

$= -\frac{(x^2 + 1)}{(x^2 - 1)^2}$

$\therefore \; f'(x)$ is negative for all $x \neq \pm 1$.

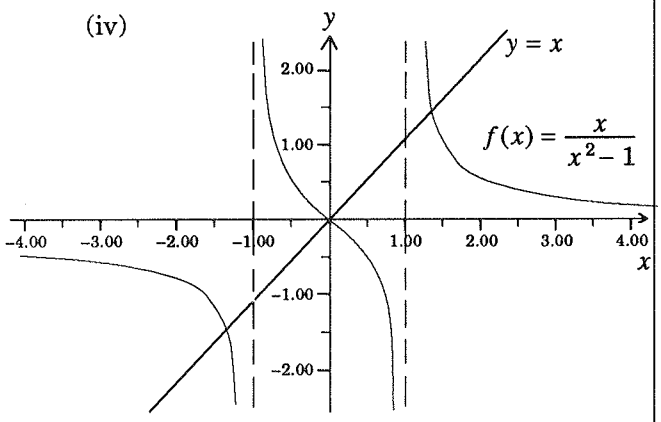
If $x = 0, \quad y = 0$

As $x \to 1^-$ $y \to -\infty$. As $x \to 1^+, y \to \infty$

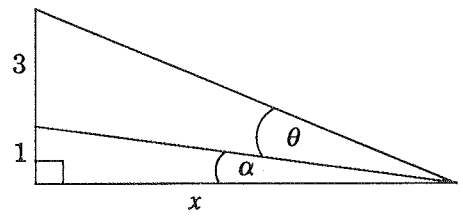
As $x \to -1^+$ $y \to \infty$. As $x \to -1^-, y \to \infty$

As $x \to \infty$ $y \to 0^+$. As $x \to = \infty, y \to 0^-$

(iv)



(b)



(i) $\tan(\theta + \alpha) = \dfrac{4}{x}$

$\therefore \quad \theta + \alpha = \tan^{-1}\dfrac{4}{x}$ ...(1)

Also $\tan\alpha = \dfrac{1}{x}$

$\therefore \quad \alpha = \tan^{-1} x$ ...(2)

Subtract (2) from (1):

$\therefore \quad \theta = \tan^{-1} x - \tan^{-1}\dfrac{1}{x}$

(ii) $$\frac{d\theta}{dx} = \frac{1}{1+\frac{16}{x^2}} \times \frac{-4}{x^2} + \frac{1}{1+\frac{1}{x^2}} \times \frac{1}{x^2}$$

$$= \frac{-4}{x^2+16} + \frac{1}{x^2+1}$$

If $\dfrac{d\theta}{dx} = \dfrac{-4x^2 - 4 + x^2 + 16}{(x^2+1)(x^2+16)} = 0$

$\therefore \quad 3x^2 = 12$

$x = \pm 2$

$\therefore$ stationary point at $x = 2$ is the relevant one.

$$f'(1) = \frac{-4}{17} + 1 > 0$$

$$f'(3) = \frac{-4}{25} + \frac{1}{10} < 0$$

$\therefore$ maximum turning point at $x = 2$.

(iii) At $\quad x = 2, \quad \theta = \tan^{-1} 2 - \tan^{-1}\dfrac{1}{2}$

$= \beta - \gamma$

Let $\tan^{-1} 2 = \beta$ and $\tan^{-1}\dfrac{1}{2} = \gamma$

$\therefore \quad \tan\beta = 2, \quad \tan\gamma = \dfrac{1}{2}$

Also $\tan(\beta - \gamma) = \tan\theta$

$$= \frac{\tan\beta - \tan\gamma}{1 + \tan\beta\tan\gamma}$$

$$= \frac{2 - \frac{1}{2}}{1+1} = \frac{3}{4}$$

$\therefore \quad \theta = \tan^{-1}\dfrac{3}{4}$