HIGHER SCHOOL CERTIFICATE MATHEMATICS
1990
3/4 UNIT SOLUTIONS

QUESTION ONE

(a) (i)
$$\int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1}x\right]_0^1 = \frac{\pi}{4}.$$

(ii)
$$\int_0^1 \frac{x}{\sqrt{1+x}}\,dx = \int_1^2 \frac{u-1}{\sqrt{u}}\,du$$
$$= \left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^2$$
$$= \frac{2}{3}\left(2-\sqrt{2}\right).$$

(b) If $x > 0$ then we want $x^2 - 4 > 0$, so $x > 2$.

If $x < 0$ then we want $x^2 - 4 < 0$,

so $-2 < x < 0$.

$\therefore$ solution is $-2 < x < 0$ or $x > 2$.

OR Multiply through by x^2 to get $x(x^2 - 4) > 0$.

The cubic is zero when $x = 0, -2, 2$, and so by sketching the cubic $y = x(x^2 - 4)$ or by testing values in the four regions $x < -2$, $-2 < x < 0$, $0 < x < 2$, $x > 2$, we find $-2 < x < 0$ or $x > 2$.

(c) (i) At the point P
$$x^2 = (x-2)^2$$
$$x^2 = x^2 - 4x + 4$$
$$x = 1.$$
Thus P has coordinates $(1, 1)$.

(ii) The tangents have gradients

$m_1 = \dfrac{d}{dx}(x^2) = 2x$, when $x = 1$, and

$m_2 = \dfrac{d}{dx}(x-2)^2 = 2x - 4$, when $x = 1$.

Thus $m_1 = 2$ and $m_2 = -2$.

If θ is the required angle,

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \frac{4}{3}$$

so $\theta = 53°$ to the nearest degree.

QUESTION TWO

(a) (i) $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

(ii) $\dfrac{2\sin^3 A + 2\cos^3 A}{\sin A + \cos A}$

$$= \frac{2(\sin A + \cos A)(\sin^2 A - \sin A\cos A + \cos^2 A)}{\sin A + \cos A}$$
$$= 2(1 - \sin A\cos A)$$
$$= 2 - 2\sin A\cos A$$
$$= 2 - \sin 2A.$$

(b) If $(x+2)$ is a factor of $p(x)$ then $p(-2) = 0$.

Thus $0 = -8 + 4a - 2b - 18$.

When $x = 1$ then $p(x)$ is -24, so

$-24 = 1 + a + b - 18$.

Thus $4a - 2b = 26$

$a + b = -7$.

Solving simultaneously gives $a = 2$ and $b = -9$.

(c) (i) The height is a maximum when $\dfrac{dy}{dt} = 0$, that is, when

$$\dot{y} = V\sin\alpha - 10t = 0$$
$$t = \frac{V\sin\alpha}{10}$$
$$\therefore\ y_{\max} = V\left(\frac{V\sin\alpha}{10}\right)\sin\alpha - 5\left(\frac{V\sin\alpha}{10}\right)^2$$
$$= \frac{V^2\sin^2\alpha}{20}.$$

(ii) The range is the x–value corresponding to $y = 0$.

When $y = 0$, $t(V\sin\alpha - 5t) = 0$,

so $$t = \frac{V\sin\alpha}{5}.$$

Thus the range is

$$R = V\left(\frac{V\sin\alpha}{5}\right).\cos\alpha$$

$$= \frac{V^2\sin\alpha\cos\alpha}{5}$$

$$= \frac{V^2\sin(2\alpha)}{10}.$$

(iii) The range is a maximum when $\sin(2\alpha) = 1$, that is, when $\alpha = 45°$.

OR For a maximum we require

$$\frac{dR}{d\alpha} = \frac{V^2\cos(2\alpha)}{5} = 0.$$

Thus $\cos(2\alpha) = 0$,

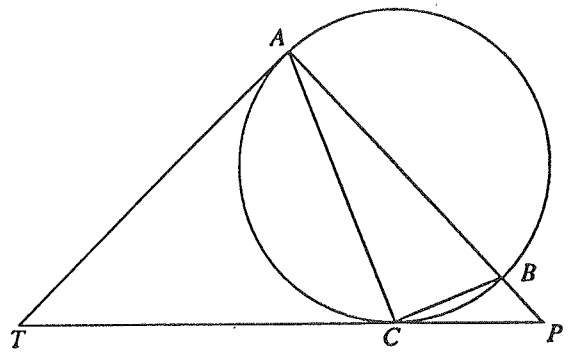
so $\alpha = 45°$ since $0 \le \alpha \le 90°$.

When $\alpha < 45°$, $\frac{dR}{dt} > 0$ and

when $\alpha > 45°$, $\frac{dR}{dt} < 0$

$\therefore$ $\alpha = 45°$ gives the maximum range.

QUESTION THREE



(a) (i) $\angle TAB = 90°$ (tangent to a circle is perpendicular to the diameter)

$\angle BCP = \angle BAC$ (angle between a tangent and a chord equals the angle in the alternate segment)

$\therefore$ $\angle TAC = 90° - \angle BAC$

$= 90° - \angle BCP.$

OR $\angle ACB = 90°$ (angle in a semicircle)

$AT = TC$ (tangents from an exterior point are of equal length)

$\therefore$ ΔTAC is isosceles

$\angle TAC = \angle ACT$ (base angles of an isosceles triangle)

$= 90° - \angle BCP$ as $\angle TCP = 180°$.

(ii) $AT = TC$ (tangents from an exterior point are of equal length)

Thus $\angle TAC = \angle TCA$ (ΔTAC is isosceles)

$\therefore$ $\angle ATC = 180° - 2(\angle TAC)$

$= 2\angle BCP.$

(b) (i) $v^2 = -5 + 6x - x^2.$ ...❶

At the end points $v = 0$, so

$v^2 = 0 = -(x - 5)(x - 1).$

The particle oscillates between 1 and 5.

OR $v^2 = -(x^2 - 6x + 9 - 4)$

$= -\{(x - 3)^2 - 4\}.$ ...❷

So the end points are $x = 3 \pm 2$,

i.e. $x = 1$ and $x = 5$.

(ii) The centre of motion of the particle is

$$x = \frac{1+5}{2} = 3.$$

(iii) The maximum speed occurs at the centre of the motion. Substituting $x = 3$ into ❶ gives

$v^2 = 4,$

i.e. maximum speed is 2 m/s;

OR from ❷, the maximum value of v^2 is 4, giving 2 m/s as the maximum speed.

(iv) The acceleration is

$$\frac{d}{dx}\left(\tfrac{1}{2}v^2\right)=\tfrac{1}{2}(6-2x)=(3-x).$$

(c) From Newton's method

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

where $f(x)=x^3-50$ and $f'(x)=3x^2$.

If $x_1 = 3\cdot5$ then

$$x_2 = 3\cdot5 - \frac{-7\cdot125}{36\cdot75}$$
$$= 3\cdot69.$$

QUESTION FOUR

(a) (i) If $u=\ln 2x$ then $\frac{du}{dx}=\frac{1}{2x}.2=\frac{1}{x}$.

$$\therefore \int \frac{\ln 2x}{x}dx = \int \ln(2x).\frac{1}{x}dx$$
$$= \int u\,du$$
$$= \frac{1}{2}u^2 + C$$
$$= \frac{1}{2}(\ln 2x)^2 + C.$$

(ii)

$$\int \cos^2 2x\,dx = \int \tfrac{1}{2}(1+\cos 4x)dx$$
$$= \tfrac{1}{2}\left(x+\frac{\sin 4x}{4}\right)+C.$$

(b) (i) Consider the number of ways of placing the four yellow marbles in the row of seven.

There are $\binom{7}{4}=\frac{7!}{4!3!}=35$ different possible arrangements of the marbles.

OR Considering the three blue marbles, we have $\binom{7}{3}=35$ arrangements.

(ii) If we only consider five of the seven marbles we need to consider the different compositions of sets of size 5.

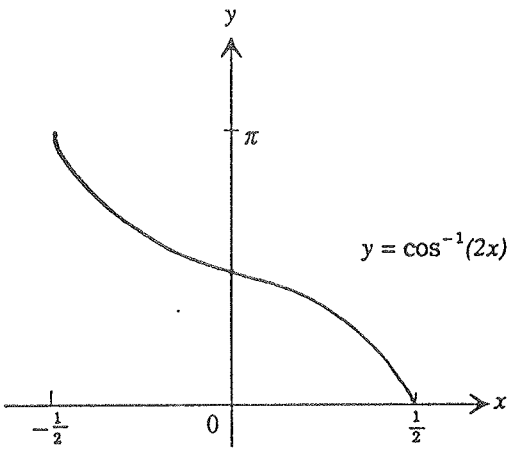
SET	Number of Arrangements
4 yellow, 1 blue	$\frac{5!}{4!}=5$
3 yellow, 2 blue	$\frac{5!}{3!2}=10$
2 yellow, 3 blue	$\frac{5!}{3!2!}=10$

Thus there are 25 possible arrangements.

(c) (i) Domain: $-1 < 2x < 1$, i.e. $-\frac{1}{2}\le x\le\frac{1}{2}$.

Range: $0 \le y \le \pi$.

(ii)



(iii)

$$y=\cos^{-1}(2x)$$
$$\frac{dy}{dx}=\frac{-1}{\sqrt{1+(2x)^2}}.2$$
$$=\frac{-2}{\sqrt{1+4x^2}}.$$

Where $x=0$, $\frac{dy}{dx}=-2$.

OR $x=\frac{1}{2}\cos y$

so $\frac{dx}{dy}=-\frac{1}{2}\sin y$.

When $x=0$, $y=\frac{\pi}{2}$,

so $\frac{dx}{dy}=-\frac{1}{2}$

and hence $\frac{dy}{dx}=-2$.

QUESTION FIVE

(a) Let $\sin\theta + \cos\theta = R\sin(\theta + \alpha)$

$= R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

$\therefore \quad R\cos\alpha = 1, \ R\sin\alpha = 1$

i.e. $R^2(\cos^2\alpha + \sin^2\alpha) = 2$

$R = \sqrt{2}$

and $\cos\alpha = \frac{1}{\sqrt{2}}, \ \sin\alpha = \frac{1}{\sqrt{2}}$

i.e. $\alpha = \frac{\pi}{4}$.

$\therefore \ \sin\theta + \cos\theta = \sqrt{2}\sin(\theta + \frac{\pi}{4})$

so $\sin\theta + \cos\theta = 1$ is equivalent to

$\sin(\theta + \frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$(\theta + \frac{\pi}{4}) = \frac{\pi}{4}, \ \frac{3\pi}{4}, \ \frac{9\pi}{4}$.

$\theta = 0, \ \frac{\pi}{2}, \ 2\pi$.

OR

If $\sin\theta + \cos\theta = 1$ then squaring both sides gives

$\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 1$

i.e. $2\sin\theta\cos\theta = 0$

$\sin 2\theta = 0$.

Thus possible values of θ are $0, \ \frac{\pi}{2}, \ \pi, \ \frac{3\pi}{2}, \ 2\pi$.

Checking each of these values we find that the solutions to the original equation in $0 \le \theta \le 2\pi$ are $0, \ \frac{\pi}{2}$, and 2π.

OR Sketch $y = \sin\theta$ and $y = 1 - \cos\theta$ and look for points of intersection.

(b) (i) If $y = mx$ intersects the circle then

$x^2 + (mx)^2 - 2x - 14mx + 25 = 0$

$(1 + m^2)x^2 - x(2 + 14m) + 25 = 0$.

If there are two distinct points in intersection, then the above quadratic must have distinct roots. Thus the discriminant must be positive, i.e.

$(2 + 14m)^2 - 4 \times 25 \times (1 + m^2) > 0$.

Dividing throughout by 4 we get,

$(1 + 7m)^2 - 25 \times (1 + m^2) > 0$.

(ii) The line $y = mx$ is a tangent to the circle when there is only one point of intersection, that is, when the discriminant above is 0.

Thus we want

$(1 + 7m)^2 - 25(1 + m^2) = 0$

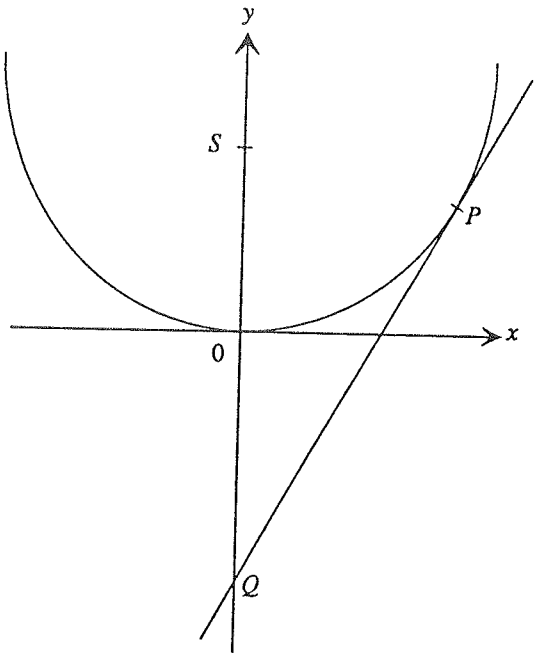
$24m^2 + 14m - 24 = 0$

$12m^2 + 7m - 12 = 0$

$(4m - 3)(3m + 4) = 0$

$m = -\frac{4}{3}, \ \frac{3}{4}$.

(c)



(i) $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

$\therefore$ The tangent to the parabola at P has slope $\frac{2ap}{2a} = p$, so the equation of the tangent is

$\frac{(y - ap^2)}{x - 2ap} = p$

$y = ap^2 + p(x - 2ap)$.

When $x = 0, \ y = -ap^2$

so Q has coordinates $(0, -ap^2)$.

(ii) Now $SQ = a + ap^2 = a(1+p^2)$

and $SP = \sqrt{(2ap)^2 + (a-ap^2)^2}$

$= \sqrt{4a^2p^2 + a^2 + a^2p^4 - 2a^2p^2}$

$= \sqrt{(a+ap^2)^2}$

$= SQ.$

(iii) The triangle SPQ is isosceles so

$180° = \angle PSQ + \angle SQP + \angle SPQ$

$= \angle PSQ + 2\angle SQP$, as $\angle SPQ = \angle SQP$.

QUESTION SIX

(a) For each question the probability of a correct answer is $\frac{1}{4}$. For 10 questions the probability of exactly 6 correct answers is

$$\binom{10}{6}\left(\frac{1}{4}\right)^6\left(\frac{3}{4}\right)^4 = 0.0162.$$

(b) (i) If $T = A + Ce^{kt}$ then

$$\frac{dT}{dt} = kCe^{kt}$$

$$= k(T-A)$$

so $T = A + Ce^{kt}$ is a solution of the differential equation.

OR $\frac{dt}{dT} = \frac{1}{k(T-A)}$

$$t = \frac{1}{k}\ln(T-A) + K$$

$$\ln(T-A) = k(t-K)$$

$$T - A = e^{k(t-K)}$$

$$= Ce^{kt}, \text{ where } C = e^{-kK}.$$

Thus $T = A + Ce^{kt}$.

(ii) Here $A = 25$, and when $t = 0$, $T = 5$,

$\therefore\ 5 = 25 + Ce^0$

so $C = -20$.

Also, when $t = 20$, $T = 10$

$\therefore\ 10 = 25 - 20e^{k20}$

$$e^{20k} = \frac{3}{4}$$

$$k = \frac{1}{20}\ln(0.75) \approx -0.01438...$$

After a further 40 minutes, that is, when t = 60,

$T = 25 - 20e^{60k}$

$= 25 - 20e^{3\ln(0.75)}$

$= 16.5625.$

The temperature will be 17° to the nearest degree.

OR $T = 25 - 20e^{60k}$

$= 25 - 20\left(e^{20k}\right)^3$

$= 25 - 20\left(\frac{3}{4}\right)^3$

$= 16.5625$

$\approx 17.$

(iii) Since $T = 25 - 20e^{-0.0144t}$, as t becomes large, $e^{-0.0144t}$ approaches 0 and so T approaches 25.

(c) (i) $x^n(1+x)^n\left(1+\frac{1}{x}\right)^n$

$$= (1+x)^n\left[x\left(1+\frac{1}{x}\right)\right]^n$$

$$= (1+x)^n(x+1)^n$$

$$= (1+x)^{2n}.$$

(ii) In the expansion of $(1+x)^{2n}$, the coefficient of x^n is $\binom{2n}{n}$.

In the expansion of $x^n(1+x)^n\left(1+\frac{1}{x}\right)^n$ the coefficient of x^n is the term independent of x in the expansion of $(1+x)^n\left(1+\frac{1}{x}\right)^n$.

Now

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + ... + \binom{n}{n}x^n$$

and $\left(1+\frac{1}{x}\right)^n$

$$= \binom{n}{0} + \binom{n}{1}\frac{1}{x} + \binom{n}{2}\left(\frac{1}{x}\right)^2 + ... + \binom{n}{n}\left(\frac{1}{x}\right)^n.$$

Thus the constant term is

$$\binom{n}{0}^2 + \binom{n}{1}^2 x.\frac{1}{x} + \binom{n}{2}^2 x^2.\frac{1}{x^2} + ... + \binom{n}{n}^2 x^n.\frac{1}{x^n}$$

and since $\binom{n}{0} = 1$, we have

$$1 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}.$$

QUESTION SEVEN

(a) Test $n = 1$ case.

$13 \times 6^1 + 2 = 80$, which is divisible by 5, so the assertion is true for $n = 1$.

Assume true for $n = k$, so assume

$13 \times 6^k + 2 = 5m$, for some integer m.

Then if $n = k + 1$

$$\begin{aligned} 13 \times 6^{k+1} + 2 &= 6(13 \times 6^k + 2) - 12 + 2 \\ &= 6 \times 5m - 10 \\ &= 5(6m - 2), \end{aligned}$$

which is divisible by 5 if the assumption holds.

OR If $T_k = 13 \times 6^k + 2$ then

$$\begin{aligned} T_{k+1} - T_k &= 13 \times 6^{k+1} - 13 \times 6^k \\ &= 13 \times 6^k (6 - 1), \end{aligned}$$

and so is divisible by 5.

Thus if T_k is divisible by 5, then so is T_{k+1}.

Hence if the result is true for $n = k$ then it is true for $n = k + 1$. Since we have established the result for $n = 1$, it follows that the result must hold for $n = 1 + 1 = 2$, $n = 2 + 1 = 3$, and so on. Thus $13 \times 6^n + 2$ is divisible by 5 for every positive integer n.

(b) (i) Using the sine rule in ΔABC

$$\frac{AC}{\sin\theta} = \frac{k}{\sin\alpha} = \frac{CB}{\sin(\pi - \theta - \alpha)}.$$

$$\therefore \quad AC = \frac{k}{\sin\alpha}.\sin\theta$$

$$CB = \frac{k}{\sin\alpha}.\sin(\pi + \theta + \alpha).$$

Thus $\ell = AC + CB$

$$= \frac{k}{\sin\alpha}[\sin\theta + \sin(\theta + \alpha)],$$

since $\sin(\pi - \theta - \alpha) = \sin(\theta + \alpha)$

(ii) α is a constant because k is constant and the angle subtended by a chord of fixed length is the same no matter where the point C is placed on the circle.

(iii) $$\frac{d\ell}{d\theta} = \frac{k}{\sin\alpha}[\cos\theta + \cos(\theta + \alpha)]$$

At $\theta = \frac{\pi}{2} - \frac{\alpha}{2}$

$$\begin{aligned} \frac{d\ell}{d\theta} &= \frac{k}{\sin\alpha}\left[\cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + \cos\left(\frac{\pi}{2} + \frac{\alpha}{2}\right)\right] \\ &= \frac{k}{\sin\alpha}\left[\sin\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\right] \\ &= 0. \end{aligned}$$

(iv) $$\frac{d^2\ell}{d\theta^2} = \frac{k}{\sin\alpha}[-\sin\theta - \sin(\theta + \alpha)] = -\ell$$

and since ℓ is a length we have that $\frac{d^2\ell}{d\theta^2} < 0$.

Thus ℓ is a maximum when $\frac{d\ell}{d\theta} = 0$,

i.e. when $\theta = \frac{\pi}{2} - \frac{\alpha}{2}$.

OR

$$\begin{aligned} \frac{d^2\ell}{d\theta^2} &= \frac{-k}{\sin\alpha}\left[\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + \sin\left(\frac{\pi}{2} + \frac{\alpha}{2}\right)\right] \\ &= \frac{-2k}{\sin\alpha}\cos\left(\frac{\alpha}{2}\right) \\ &< 0 \text{ since } 0 \le \frac{\alpha}{2} \le \frac{\pi}{2}. \end{aligned}$$

OR Write

$$\ell = \frac{k}{\sin\alpha}\left[2\sin\left(\frac{\theta + (\theta + \alpha)}{2}\right).\cos\left(\frac{(\theta + \alpha) - \theta}{2}\right)\right]$$

$$= \frac{2k\cos(\alpha/2)}{\sin\alpha}.\sin(\theta + \alpha/2),$$

which is clearly maximized when $\theta + \frac{\alpha}{2} = \frac{\pi}{2}$

i.e. $\theta = \frac{\pi}{2} - \frac{\alpha}{2}$.