# HIGHER SCHOOL CERTIFICATE EXAMINATION 1989
## MATHEMATICS - 3/4 UNIT COMMON PAPER

### QUESTION ONE

(a) $2m^3 - 128 = 2(m^3 - 64)$

$= 2(m-4)(m^2 + 4m + 16)$

(b) (i) $\displaystyle\int_1^2 \frac{1}{\sqrt{4-x^2}}\,dx = \left[\sin^{-1}\frac{x}{2}\right]_1^2$

$= \sin^{-1}1 - \sin^{-1}\frac{1}{2}$

$= \frac{\pi}{2} - \frac{\pi}{6}$

$= \frac{\pi}{3}$

NB: $\frac{1}{\sqrt{4-x^2}}$ *is not defined for* $x = 2$ *but the integral can be evaluated.*

(b) (ii) $\displaystyle\int_{-1}^0 x\sqrt{1+x}\,dx$ Put $u = 1 + x$

$x = -1\ ,\ u = 0$

$x = 0\ ,\ u = 1$

$= \displaystyle\int_0^1 (u-1)\,u^{1/2}\,du$

$= \displaystyle\int_0^1 \left(u^{3/2} - u^{1/2}\right)du$

$= \left[\frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3}\right]_0^1$

$= \frac{2}{5} - \frac{2}{3}$

$= -\frac{4}{15}$

(c) A(1, 4) B(5, 2) -1:3

$\left(\frac{-1\times 5 + 3\times 1}{-1+3}, \frac{-1\times 2 + 3\times 4}{-1+3}\right)$

$= (-1, 5)$

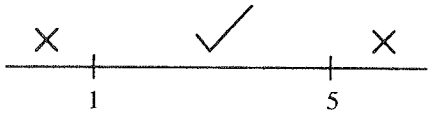
(d) $\frac{4}{5-x} \geq 1$

$x \neq 5$

Solve $\frac{4}{5-x} = 1$

$\therefore\ 4 = 5 - x$

$x = 1$




Test $x < 1$ say $x = 0$, LHS $= {}^4/_5 < 1$, False

$1 < x < 5$ say $x = 2$, LHS $= {}^4/_3 \geq 1$, True

$x > 5$ say $x = 6$, LHS $= {}^4/_{-1} < 1$, False

$\therefore\quad 1 \leq x < 5$

*Alternatively*

$\frac{4}{5-x} \geq 1$

$\therefore (5-x)^2 \times \frac{4}{5-x} \geq (5-x)^2$

$20 - 4x \geq 25 - 10x + x^2$

$\therefore x^2 - 6x + 5 \leq 0$

$(x-5)(x-1) \leq 0$

$1 \leq x \leq 5$

But $x \neq 5$

$\therefore\quad 1 \leq x < 5$

*Alternatively*

If $5 - x > 0$ then $4 \geq 5 - x$

*i.e.* $x \geq 1$ and $x < 5$

If $5 - x < 0$ then $4 \leq 5 - x$

*i.e.* $x \leq 1$ and $x > 5$

$\therefore$ No solution

Hence, the only solution is $1 \leq x < 5$

*Alternatively*

Graphs of $y = \frac{4}{5-x}$ and $y = 1$ intersect at $y = 1, x = 1$ (1,1)

Hence $\frac{4}{5-x} \geq 1$ *when* $1 \leq x \leq 5$

## QUESTION TWO

(a) Since $PQ^2 = PR.PS$

$$6^2 = x(x+5)$$
$$= x^2 + 5x$$
$$\therefore x^2 + 5x - 36 = 0$$
$$(x+9)(x-4) = 0$$
$$x = -9, 4$$
$\therefore$ Since $x \geq 0$, $x = 4$

(b) $0 \leq \theta \leq 2\pi$

$$\sin 2\theta = \sin\theta$$
$$2\sin\theta\cos\theta = \sin\theta$$
$$2\sin\theta\cos\theta - \sin\theta = 0$$
$$\sin\theta(2\cos\theta - 1) = 0$$
$$\sin\theta = 0 \text{ or } \cos\theta = \frac{1}{2}$$
$$\therefore \theta = 0, \pi, 2\pi \text{ or } \frac{\pi}{3}, \frac{5\pi}{3}$$

(c) (i) $v = \frac{dx}{dt}$

Now $\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt}$ (chain rule)

$$= \frac{dv}{dx} \cdot v \quad ...(1)$$

Also $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dv}\left(\frac{1}{2}v^2\right) \times \frac{dv}{dx}$

$$= v \cdot \frac{dv}{dx}$$
$$= \frac{dv}{dt} \text{ from } ...(1)$$

(ii) $\ddot{x} = -2e^{-x}$

$$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -2e^{-x}$$
$$\therefore \frac{1}{2}v^2 = +2e^{-x} + C$$

When $t = 0$, $x = 0$, $v = 2$

$$\therefore \frac{1}{2}2^2 = +2e^0 + C$$
$$2 = 2 + C$$
$$\therefore C = 0$$
$$\therefore \frac{1}{2}v^2 = 2e^{-x}$$
$$v^2 = 4e^{-x}$$
$$v = \pm 2e^{-x/2}$$

However since $v = 2$ (>0) when $x = 0$ then the positive value must be chosen

$$\therefore \quad v = 2e^{-x/2}$$

(iii) As $x \to \infty$, $e^{-x/2} \to 0$

$$\therefore \quad v \to 0$$

Velocity approaches zero.

## QUESTION THREE

(a) (i) Number of Committees $= {}^8C_3$ or $\binom{8}{3}$

$$= \frac{8 \times 7 \times 6}{3 \times 2 \times 1}$$
$$= 56$$

(a) (ii) Number of ways of getting a committee of 3 from 4 Queenslanders

$$= {}^4C_3 = 4$$

$\therefore$ The probability of only Queenslanders

$$= \frac{4}{56}$$
$$= \frac{1}{14}$$

(b) $\left(x - \frac{1}{2x^3}\right)^{20}$

The general term is given by

$$T_{n+1} = {}^{20}C_n x^{20-n}\left(-\frac{1}{2x^3}\right)^n$$

$$= {}^{20}C_n x^{20-n}\left(-\frac{1}{2}\right)^n . x^{-3n}$$

For a constant term $20 - n - 3n = 0$

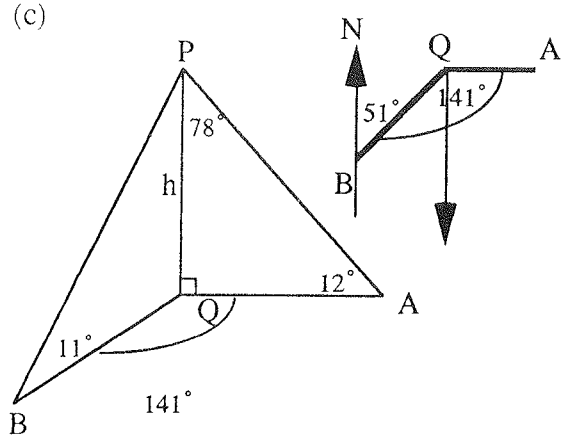
$$4n = 20$$

$$n = 5$$

$\therefore$ Term is ${}^{20}C_5\left(-\frac{1}{2}\right)^5$

$= -484.5$

(c)




(i) $\angle AQB = 90° + 51°$

Since QA and QS are at right angles

$\angle NBQ = \angle BQS$ (alt angles, NB|| QS)

Since $\angle NBQ = 51°$ (Q is 051° from B)

$\therefore \angle AQB = 141°$

(ii) In $\Delta APQ$, $\angle APQ = 78°$

(angle sum of $\Delta APQ$)

$$\frac{AQ}{h} = \tan 78°$$

$\therefore AQ = h \tan 78°$

(iii) In $\Delta PQB$, $\angle BPQ = 79°$

$BQ = h \tan 79°$

(iv) In $\Delta ABQ$,

$$AB^2 = AQ^2 + BQ^2 - 2.AQ.\ BQ.\cos 141°$$

$$\therefore 1\,000\,000 = h^2 \tan^2 78° + h^2 \tan^2 79° - 2h^2 \tan 78°.\tan 79°.\cos 141°$$

$$\therefore h^2 = \frac{1\,000\,000}{\tan^2 78° + \tan^2 79° - 2\tan 78°.\tan 79°.\cos 141°}$$

$$h^2 \approx \frac{1\,000\,000}{86.2188\ldots}$$

$h \approx 107.6958\ldots$

$h = 108$m (to the nearest metre)

## QUESTION FOUR

(a) $A = \pi r^2$ for any circle , radius $r$.

$\therefore \frac{dA}{dr} = 2\pi r$ and $\frac{dA}{dt} = 3.2$, given

Since $\frac{dA}{dr} \times \frac{dr}{dt} = \frac{dA}{dt}$

$$\frac{dr}{dt} = \frac{dA}{dt} \div \frac{dA}{dr} \quad \text{when } r = 10$$

$$= \frac{3.2}{2 \times \pi \times 10}$$

$$= 0.050929$$

Radius increases at 0.051 cm $\text{min}^{-1}$ when $r = 10$

(b) (i) $P(x) = (x-a)^2 Q(x)$

$$P'(x) = (x-a)^2 Q'(x) + 2(x-a)Q(x)$$

$$= (x-a)[(x-a)Q'(x) + 2Q(x)]$$

$$P'(a) = (a-a)[(a-a)Q'(a) + 2Q(a)]$$

$$= 0$$

(ii) If $x = 1$ is a double root of

$P(x) = x^4 + ax^3 + bx^2 - 5x + 1 = 0$

then $P(1) = 0$

$\therefore 1 + a + b - 5 + 1 = 0$

*i.e.* $a + b = 3$ (1)

$P'(x) = 4x^3 + 3ax^2 + 2bx - 5$

and from (i) $P'(1) = 0$

$\therefore 4 + 3a + 2b - 5 = 0$

*i.e.* $3a + 2b = 1$

from (1) $3a + 3b = 9$

$\therefore b = 8$

and $a = -5$

*Alternatively*

When P($x$) is divided by $x - 1$ the remainder is zero.

By long division

$x - 1 \overline{)x^4 + ax^3 + bx^2 - 5x + 1}$

has a quotient of

$x^3 + (a + 1)x^2 + (b + a + 1)x + a + b - 4$

and a remainder of $1 + a + b - 4 = 0$

$\therefore a + b = 3$ ____(i)

Since $(x - 1)$ was a double root, a second division of the above quotient by $x - 1$ gives a remainder which simplifies to

$3a + 2b - 1 = 0$ ____(ii)

Solving(i) and (ii) gives a = -5, b = 8 as before.

*Alternatively*

Since $(x - 1)$ is a double root

$P(x) = (x - 1)^2(x^2 + qx + 1)$

$= x^4 + (q - 2)x^3 + (2 - 2q)x^2 + (q - 2)x + 1$

Equating coefficients $q - 2 = -5$

$\therefore q = -3$

$\therefore a = q - 2 = -5$

$b = 2 - 2q = 2 + 6 = 8$

*Alternatively*

Let the roots be $1, 1, \alpha, \beta$

Then $\Sigma\alpha$ : $2 + \alpha + \beta = -a$

$\Sigma\alpha\beta$ : $1 + 2\alpha + 2\beta + \alpha\beta = b$

$\Sigma\alpha\beta\gamma$ : $\alpha + \beta + 2\alpha\beta = 5$

$\alpha\beta\gamma\delta$ : $\alpha\beta = 1$

Since $\alpha\beta = 1$, substitute in $\alpha + \beta + 2\alpha\beta = 5$

$\therefore \alpha + \beta = 3$

$\therefore 2 + 3 = -a, \quad \therefore a = -5$

$1 + 2\alpha + 2\beta + \alpha\beta = b$

$\therefore b = 1 + 6 + 1 = 8$

(c) D E L E T E D

has 7 Letters, 2 D'S and 3 E'S

(i) Number of words $= \frac{7!}{3! \times 2!}$

$= 420$

(ii) Place the D's together. There are 6 separate things to be arranged, containing 3 E's.

The number of ways $= \frac{6!}{3!}$

$= 120$

Hence the number of ways with the D's separated is

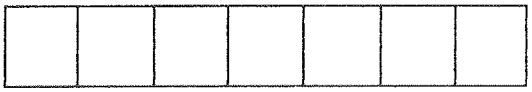
total number – number with D's together

$= 420 - 120$

$= 300$

*Alternatively*

Consider 7 places to be filled



We can put D's one apart in 5 ways

two apart in 4 ways

three apart in 3 ways

four apart in 2 ways

five apart in 1 way

So number of ways = 5 + 4 + 3 + 2 + 1 = 15

The other 5 letters can be arranged in

$\frac{5!}{3!}$ ways $= 20$

$\therefore$ No. of arrangements = 15 × 20

$= 300$

*Alternatively*

Consider a D placed at an end: this can be done 2 ways. The next position can be filled 5 ways, and the remaining places 5! ways.

i.e. No. of ways is $2 \times 5 \times \frac{5!}{2!3!} = 100$

Consider next a D places in a non-end position, this can be done 5 ways. Positions on either side can be filled 5 ways and 4 ways. The remaining places filled 4! ways.

i.e. No. of ways is $5 \times 5 \times 4 \times \frac{4!}{3!2!} = 200$

$\therefore$ The total no. of ways = 300

*Alternatively*

Consider the letters excluding the D's.

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They can occupy 5 places, and be arranged in $\frac{5!}{3!}$ ways.

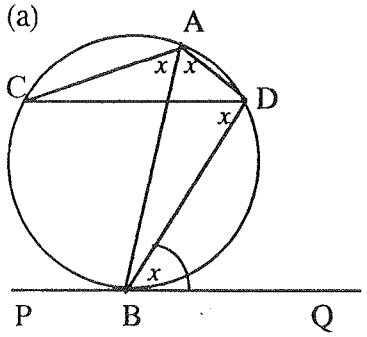
Now the two D's can be placed in any of 6 positions in the spaces between or at the ends of these other 5 letters.

The first D can be placed in 6 ways, the second D in 5 ways. But since the D's are the same, division by 2! is needed.

$\therefore$ The number of ways $= \frac{6 \times 5}{2!} \times \frac{5!}{3!} = 300$

## QUESTION FIVE

(a)




Join BD

$\angle DBQ = \angle BAD$ (angle between tangent and chord equals angle in the alternate segment)

$\angle DBQ = \angle CDB$ (alt. angles, CD//BQ)

$\angle CDB = \angle CAB$ (standing on same arc BC)

$\therefore \angle CAB = \angle BAD$

$\therefore$ AB bisects $\angle CAD$

(b) (i) $1 + 2 + 3 + \ldots\ldots n$

Arithmetic sequence with a = 1, d = 1.

$$S_n = \frac{n}{2}(a + L)$$
$$= \frac{n}{2}(1 + n)$$

$$\therefore (1 + 2 + 3 + \ldots\ldots n)^2 = \left(\frac{n}{2}(1 + n)\right)^2$$
$$= \frac{1}{4}n^2(n + 1)^2$$

(ii) Assume $1^3 + 2^3 + \ldots\ldots n^3 = \frac{1}{4}n^2(n + 1)^2$

for some value of n

then $1^3 + 2^3 + \ldots\ldots n^3 + (n + 1)^3$

$$= \frac{1}{4}n^2(n + 1)^2 + (n + 1)^3$$
$$= \frac{1}{4}(n + 1)^2\left[n^2 + 4(n + 1)\right]$$
$$= \frac{1}{4}(n + 1)^2(n + 2)^2$$

This is the result with $n + 1$ for $n$.

When n = 1, LHS $= 1^3 = 1$,

$$\text{RHS} = \frac{1}{4} \cdot 1^2 \cdot 2^2 = 1$$

$\therefore$ The result is true for $n = 1$.

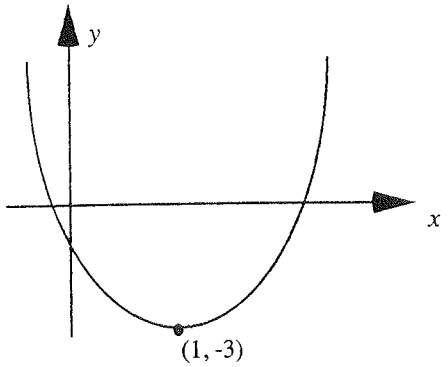
$\therefore$ The result is true for $n = 2$ and if true for $n = 2$ is true for $n = 3$.

$\therefore$ by induction it is true for all $n$

i.e. $1^3 + 2^3 + \ldots\ldots n^3 = \frac{1}{4}n^2(n + 1)^2$

$= (1 + 2 + \ldots + n)^2$ from (i)

## QUESTION SIX




$y = (x-1)^2 - 3$ or $(x-1)^2 = y + 3$

$\therefore$ Vertex is at (1, -3)

(i) If a function has an inverse it must be either monotonic increasing or decreasing but not both. For $x \geq 1$, the curve is increasing and will have an inverse.

(ii) For the inverse function

$$(y-1)^2 = x + 3$$

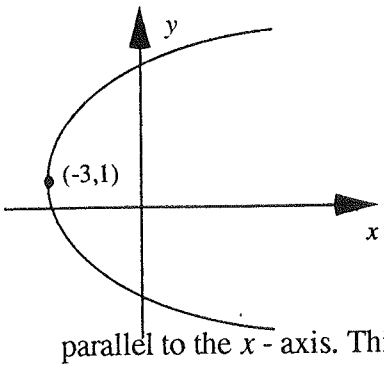
$$\therefore\ y - 1 = \sqrt{x+3}$$

$$\left(\text{since } y \geq 1,\ y - 1 \neq -\sqrt{x+3}\right)$$

$$\therefore\ y = 1 + \sqrt{x+3}$$

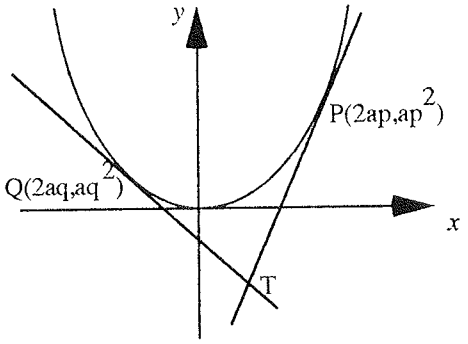
$\therefore$ Domain is $x \geq -3$ ( since $x + 3 \geq 0$ )

(iii) The function will not have an inverse if there are included values of x on both sides of $x = 1$, so no inverse for $-1 \leq x \leq 3$, say.




The function with $x$ and $y$ interchanged would become a parabola with its axis parallel to the $x$-axis. This is not a function because for each value of $x$ there are two values of $y$

(b)




Since $x^2 = 4ay$

$$\therefore\ \frac{dy}{dx} = \frac{x}{2a}$$

$$= \frac{2ap}{2a} \text{ at P}$$

$$= p$$

$\therefore$ The tangent at P$(2ap, ap^2)$ is

$$y - ap^2 = p(x - 2ap)$$

$$y = px - ap^2$$

(ii) The tangent at Q$(2aq, aq^2)$ is similarly

$$y = qx - aq^2$$

Solving simultaneously

$$px - ap^2 = qx - aq^2$$

$$px - qx = ap^2 - aq^2$$

$$(p-q)x = a(p-q)(p+q)$$

$$\therefore\ x = a(p+q) \text{ since } p \neq q$$

$$y = qx - aq^2$$

$$= apq + aq^2 - aq^2$$

$$= apq$$

$\therefore$ T is the point $(a(p+q), apq)$

(iii) $$\tan 45^\circ = \left|\frac{p-q}{1+pq}\right| = 1$$

$$\therefore\ p - q = 1 + pq, \text{ given } p > q$$

(iv) Consider $x^2 - 4ay$ at $T$

$$x^2 - 4ay = a^2(p+q)^2 - 4a^2pq$$
$$= a^2\left[p^2 + q^2 + 2pq - 4pq\right]$$
$$= a^2\left[p - q\right]^2$$
$$= a^2\left[1 + pq\right]^2 \text{ from (iii)}$$
$$= a^2\left[1 + \frac{y}{a}\right]^2 \text{ from (ii)}$$
$$= a^2 + 2ay + y^2$$

$$x^2 - y^2 - 6ay = a^2$$
$$x^2 - (y - 3a)^2 = 10a^2$$

this equation is that of a hyperbola

*Alternatively*

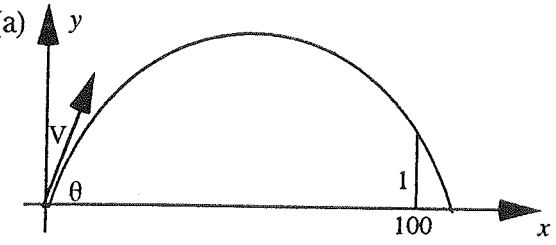
$$p - q = 1 + pq \text{ from (iii)}$$

And for T, $x = a(p+q)$

$$y = apq$$
$$\therefore y = a(p - q - 1)$$
$$= ap - aq - a$$
$$\therefore y + a = a(p - q)$$
$$(y+a)^2 = a^2(p^2 - 2pq + q^2)$$
$$= a^2\left[(p+q)^2 - 4pq\right]$$
$$= a^2\left[\frac{x^2}{a^2} - 4\frac{y}{a}\right]$$
$$= x^2 - 4ay$$

**QUESTION SEVEN**

(a)




| Vertical Motion | Horizontal Motion |
|---|---|
| $\ddot{y} = -10$ | No acceleration |
| $\dot{y} = -10t + C_1$ | $\dot{x} = V\cos\theta$, constant |
| $t = 0, \dot{y} = V\sin\theta$ | $\dot{x} = 32\cos\theta$ |
| $\therefore C_1 = 32\sin\theta$ | $x = 32t\cos\theta + C_2$ |
| $\therefore \dot{y} = -10t + 32\sin\theta$ | $t = 0, x = 0, C_2 = 0$ |
| $\therefore y = -5t^2 + 32t\sin\theta + K$ | $x = 32t\cos\theta$ |
| $t = 0, y = 0, K = 0$ | $\therefore t = \frac{x}{32\cos\theta}$ |

$$\therefore y = -5t^2 + 32t\sin\theta$$
$$= -5\left(\frac{x}{32\cos\theta}\right)^2 + 32\frac{x}{32\cos\theta}\sin\theta$$
$$= \frac{-5x^2}{1024}\sec^2\theta + x\tan\theta$$
$$= \frac{-5x^2}{1024}(1 + \tan^2\theta) + x\tan\theta$$
$$1024\,y = -5x^2 - 5x^2\tan^2\theta + 1024x\tan\theta$$

(ii) When $x = 100$, $y = 1$ since the ball just clears the boundary fence.

$$1024 = -50000 - 50000\tan^2\theta + 102400\tan\theta$$
$$\therefore 50000\tan^2\theta + 102400\tan\theta + 51024 = 0$$

(iii) For a six, then when $x = 100$, $y > 1$

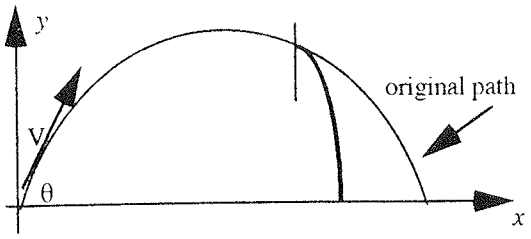
i.e. $50000\tan^2\theta + 102400\tan\theta + 51024 < 0$

$$\tan\theta = \frac{102400 \pm \sqrt{102400^2 - 4.50000.51024}}{100000}$$
$$\approx 1.1916 \text{ or } 0.8563$$

i.e. $\theta$ lies between 40°35' and 50°0'

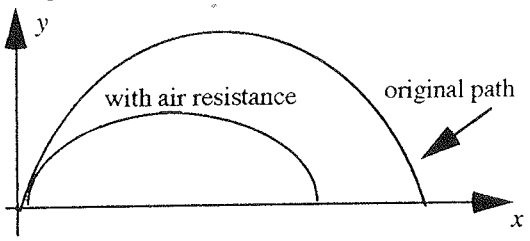
[NOTE: 45° gives the maximum range so range must include 45°]

(iv)




Ball loses horizontal velocity, so falls more steeply, but still parabolic in form.

Air resistance: both horizontal and vertical velocities are reduced. So ball will not rise so far, nor cover such a great distance with the same hit. It would be necessary to increase the initial velocity and reduce the range of θ to still score a six.




(b) (i) $y = \tan^{-1}\frac{1}{x}$ put $u = \frac{1}{x}$ $x \neq 0$

$y = \tan^{-1} u$ $\quad \frac{du}{dx} = \frac{-1}{x^2}$

$$\frac{dy}{du} = \frac{1}{1+u^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{1+\left(\frac{1}{x}\right)^2} \times \frac{-1}{x^2}$$

$$= \frac{-1}{x^2+1}$$

$$\frac{d}{dx}\left(\tan^{-1}x + \tan^{-1}\frac{1}{x}\right) = \frac{1}{1+x^2} - \frac{1}{x^2+1}$$

$$= 0$$

(b) (ii)

when $x = 1$, $y = \tan^{-1}1 + \tan^{-1}\frac{1}{1}$

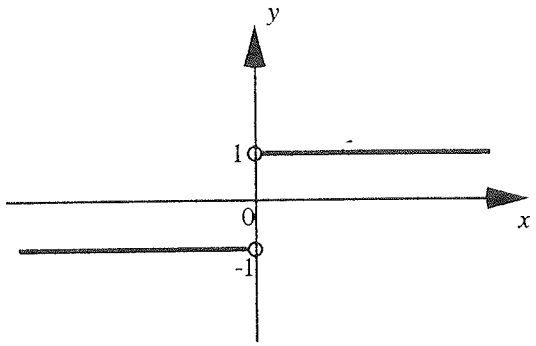
$$= \frac{\pi}{4} + \frac{\pi}{4}$$

$$= \frac{\pi}{2}$$

when $x = -1$, $y = \tan^{-1}(-1) + \tan^{-1}(-1)$

$$= \frac{-\pi}{2}$$

Function is undefined at $x = 0$




The function is odd and consists of two straight lines $y = \frac{\pi}{2}$ when $x > 0$

and $y = \frac{-\pi}{2}$ when $x < 0$.