QUESTION 1

(a) (i)

$$\frac{d}{dx}(x\cos x) = 1.\cos x + x.-\sin x$$
$$= \cos x - x\sin x$$

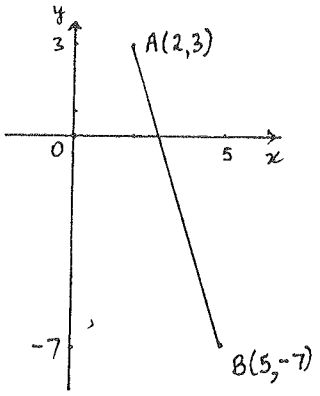
(ii)

$\frac{d}{dx}(\tan^{-1}3x)$:

Let $y = \tan^{-1}u, \quad u = 3x$

$$\frac{dy}{du} = \frac{1}{1+u^2} \qquad \frac{du}{dx} = 3$$
$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{1}{1+u^2} \times 3$$
$$= \frac{3}{1+(3x)^2}$$
$$= \frac{3}{1+9x^2}$$

(b)



$$P \equiv \left(\frac{4\times 5 + 9\times 2}{4+9}, \frac{4\times -7 + 9\times 3}{4+9}\right)$$
$$\equiv \left(\frac{20+18}{13}, \frac{-28+27}{13}\right)$$
$$\equiv (2\tfrac{12}{13}, -\tfrac{1}{13})$$

(c) (i)

$$\int_0^1 \frac{2x}{x^2+1}\,dx$$
$$= \left[\log_e(x^2+1)\right]_0^1$$
$$= \log_e 2 - \log_e 1$$
$$= \log_e 2$$

(ii)

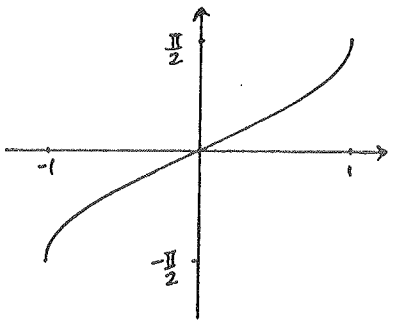
$$\int_0^{\pi} \sin^2 x\,dx$$
$$= \tfrac{1}{2}\int_0^{\pi} 1 - \cos 2x\,dx$$
$$= \tfrac{1}{2}[x - \tfrac{1}{2}\sin 2x]_0^{\pi}$$
$$= \tfrac{1}{2}\{(\pi - \tfrac{1}{2}\sin 2\pi) - (0-0)\}$$
$$= \frac{\pi}{2}$$

(d)

No. of arrangements
$$= \frac{6!}{2!} \quad \text{since there are 2 Y's}$$
$$= 360$$

QUESTION 2

(a) (i)



$$y = \sin^{-1}x$$
Domain: $\{-1 \le x \le 1\}$
Range: $\{-\frac{\pi}{2} \le y \le \frac{\pi}{2}\}$

(ii)

$$\int_0^1 \sin^{-1}x\,dx$$
$$\approx \tfrac{0.5}{3}\left[f(0) + f(1) + 4f(\tfrac{1}{2})\right]$$
$$= \tfrac{0.5}{3}\left[\sin^{-1}0 + \sin^{-1}1 + 4\sin^{-1}(\tfrac{1}{2})\right]$$
$$= \tfrac{0.5}{3}\left[0 + \frac{\pi}{2} + 4\times\frac{\pi}{6}\right]$$
$$= \tfrac{0.5}{3}\left[\frac{\pi}{2} + \frac{2\pi}{3}\right]$$
$$= \frac{0.5}{3}\times\frac{7\pi}{6}$$
$$= \frac{7\pi}{36}$$
$$\approx 0.61 \text{ (to 2 dec. places)}$$

(b) If $u = 2 + x^2$, then $du = 2x\,dx$

$$\int x\sqrt{2+x^2}\,dx$$
$$= \tfrac{1}{2}\int \sqrt{u}\,du$$
$$= \tfrac{1}{2}.\tfrac{2}{3}u^{\frac{3}{2}} + c$$
$$= \tfrac{1}{3}\sqrt{(2+x^2)^3} + c$$

(c) α, β, γ are the roots of $x^3 - 3x + 1 = 0$

(i) $\alpha + \beta + \gamma = 0$

(ii) $\alpha\beta\gamma = \frac{-1}{1} = -1$

(iii)

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$$
$$=\frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$$
$$=\frac{-3}{-1}$$
$$=3$$

QUESTION 3

(a) (i)

$$x=3\cos(2t+\tfrac{\pi}{3})$$
$$\therefore \dot{x}=3\times -2\sin(2t+\tfrac{\pi}{3})$$
$$=-6\sin(2t+\tfrac{\pi}{3})$$
$$\ddot{x}=-6\times 2\cos(2t+\tfrac{\pi}{3})$$
$$=-12\cos(2t+\tfrac{\pi}{3})$$
$$=-4\times(3\cos(2t+\tfrac{\pi}{3}))$$
$$=-4x$$

(ii) Amplitude = 3 cm

(iii) When speed is a maximum, $\ddot{x}=0$, i.e. there is no acceleration.

i.e. $-12\cos(2t+\frac{\pi}{3})=0$

i.e. $2t+\frac{\pi}{3}=\frac{\pi}{2}$

$\therefore x=3\cos(2t+\frac{\pi}{3})=3\cos\frac{\pi}{2}=0$

[NB particle exercises SHM about O]

Speed at O, the max. speed
$=|-6\sin\frac{\pi}{2}|$
$=|-6|$
$=6$ cm/sec.

(b) Assume that

$$1^2+3^2+\ldots+(2n-1)^2=\tfrac{n}{3}(2n-1)(2n+1)$$

$$\therefore 1^2+3^2+\ldots+(2n-1)^2+(2n+1)^2$$
$$=\tfrac{n}{3}(2n-1)(2n+1)+(2n+1)^2$$
$$=\tfrac{1}{3}(2n+1)[n(2n-1)+3(2n+1)]$$
$$=\tfrac{1}{3}(2n+1)[2n^2-n+6n+3]$$
$$=\tfrac{1}{3}(2n+1)(2n^2+5n+3)$$
$$=\tfrac{1}{3}(2n+1)(2n+3)(n+1)$$
$$=\frac{(n+1)}{3}(2(n+1)-1)(2(n+1)+1)$$

which has the same form as $\Sigma_1^n(2r-1)^2$, i.e. if the formula is true for some integer, n, then it is also true for the next integer, $n+1$.

The formula is true for $n=1$
(since $\frac{1}{3}(2\times 1-1)(2\times 1+1)=\frac{1}{3}\times 1\times 3=1$.)

$\therefore$ the formula is true for $n=2$ and then again for $n=3$ and so again for $n=4$ etc. forever.

$\therefore$ The formula is proven true for all integers, $n\geq 1$, by mathematical induction.

(c)

$$\tan\left(\frac{\pi}{4}+x\right)$$
$$=\frac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}.\tan x}$$
$$=\frac{1+\tan x}{1-\tan x}$$
$$=\frac{1+\frac{\sin x}{\cos x}}{1-\frac{\sin x}{\cos x}}$$
$$=\frac{\cos x+\sin x}{\cos x-\sin x}$$

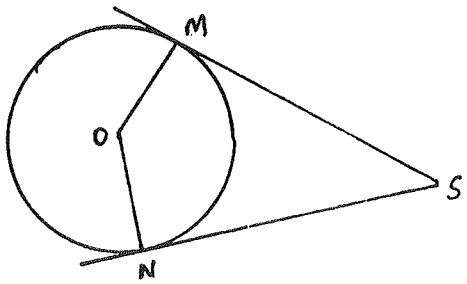
Alternately

$$\tan\left(\frac{\pi}{4}+x\right)$$
$$=\frac{\sin(\frac{\pi}{4}+x)}{\cos(\frac{\pi}{4}+x)}$$
$$=\frac{\sin\frac{\pi}{4}\cos x+\cos\frac{\pi}{4}\sin x}{\cos\frac{\pi}{4}\cos x-\sin\frac{\pi}{4}\sin x}$$
$$=\frac{\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x}{\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x}$$
$$=\frac{\cos x+\sin x}{\cos x-\sin x}$$

(after multiplying both num. & denom. by $\sqrt{2}$)

QUESTION 4

(a) (i)



$OM=ON$ (radii)

$\angle OMS=\angle ONS$

(angle between a tangent and a radius at the point of contact is a right angle)

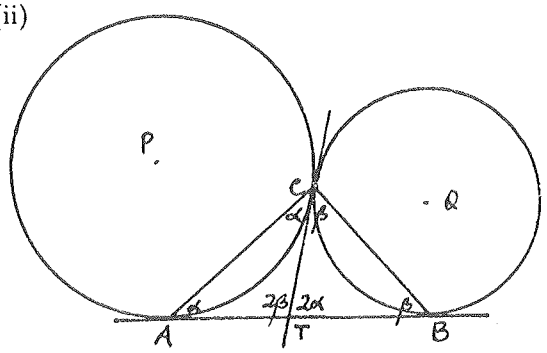
OS is common

$\therefore \triangle OMS\equiv\triangle ONS$ (RHS)

$\therefore SM=SN$

i.e. the tangents drawn to a circle from an external point are equal.

(ii)



(α)

$TA = TC$ (tangents to circle, centre P, from T)

$TB = TC$ (tangents to circle, centre Q, from T)

(β)

$TA = TC$

$\therefore \triangle TAC$ is isosceles

$\therefore \angle TAC = \angle TCA = \alpha$, say

$\therefore \angle CTB = 2\alpha$ (ext. angle of a triangle = sum of int. opp. angles)

Sim. in $\triangle TCB$: $\angle TCB = \angle TBC = \beta$, say.

and $\angle CTA = 2\beta$

$\angle ATB$ is a straight angle

$\therefore 2\alpha + 2\beta = 180°$

$\therefore \alpha + \beta = 90°$

$\therefore \angle ACT + \angle BCT = \angle ACB = 90°$

(b) (i)

$$
\begin{array}{r}
2x^2\ -4x-2 \\
x^2-3x+1\ \overline{)\ 2x^4-10x^3+12x^2+2x-3} \\
2x^4-\ 6x^3+2x^2 \\
\hline
-4x^3\ +10x^2+2x \\
-4x^3\ +12x^2-4x \\
\hline
-2x^2\ +6x-3 \\
-2x^2\ +6x-2 \\
\hline
-1
\end{array}
$$

(ii) $\therefore f(x) = g(x)q(x) + r(x)$ becomes

$2x^4 - 10x^3 + 12x^2 + 2x - 3$

$= (x^2 - 3x + 1)(2x^2 - 4x - 2) - 1$

where $q(x) = 2x^2 - 4x - 2$; $\quad r(x) = -1$

Note: $r(x)$ has degree zero.

(iii) If $f(x)$ and $g(x)$ have a common zero, say α, then $f(\alpha) = g(\alpha) = 0$.

But $f(\alpha) = g(\alpha)q(\alpha) + r(\alpha)$

so that $0 = 0.q(\alpha) + r(\alpha)$

i.e. $r(\alpha) = 0$

But $r(\alpha) = -1$

$\therefore$ there is a contradiction

$\therefore f(x)$, $g(x)$ have no common zeros.

QUESTION 5

(a) (i)

$y = x - 2\sin x \qquad 0 \le x \le 2\pi$

$y' = 1 - 2\cos x$

$y'' = 2\sin x$

Stat. pt.: $y' = 0$

i.e. $2\cos x = 1$

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}, \frac{5\pi}{3} \qquad (0 \le x \le 2\pi)$

When $x = \frac{\pi}{3}$, $y'' = 2\sin\frac{\pi}{3} > 0$

$y = \frac{\pi}{3} - 2\sin\frac{\pi}{3}$

$= \frac{\pi}{3} - 2 \times \frac{\sqrt{3}}{2}$

$= \frac{\pi}{3} - \sqrt{3}$

$\therefore$ Rel. min. pt. $(\frac{\pi}{3}, \frac{\pi}{3} - \sqrt{3})$

$\approx (\frac{\pi}{3}, -0.68)$

When $x = \frac{5\pi}{3}$, $y'' = 2\sin\frac{5\pi}{3} < 0$

$y = \frac{5\pi}{3} - 2\sin\frac{5\pi}{3}$

$= \frac{5\pi}{3} - 2 \times \frac{-\sqrt{3}}{2}$

$= \frac{5\pi}{3} + \sqrt{3}$

$\therefore$ Rel. max. pt. $(\frac{5\pi}{3}, \frac{5\pi}{3} + \sqrt{3})$

$\approx (\frac{5\pi}{3}, 6.97)$

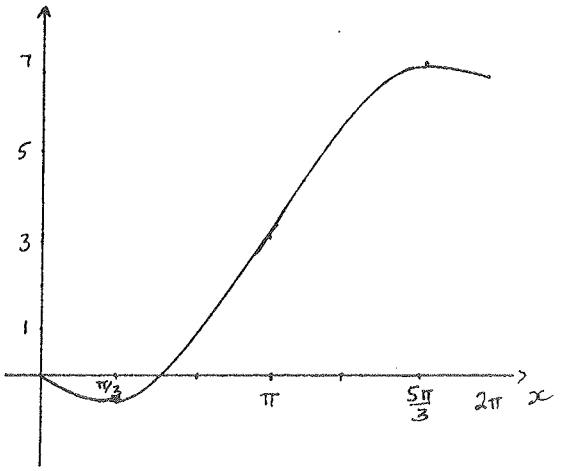
(ii)

$x = 0, \quad y = 0 - 2\sin 0 = 0$

$x = \pi, \quad y = \pi - 2\sin\pi = \pi - 0 = \pi$

$x = 2\pi, \quad y = 2\pi - 2\sin 2\pi = 2\pi$

(iii)



(b) (i) Count the three teenagers as one group. There are seven other groups (with one person per group). These 8 groups may be arranged in 8! ways around a circular table, although 8 of these are indistiguishable in terms of who is next to whom.

$\therefore$ No. of arrangements $= \frac{8!}{8} = 7!$

Each of these can allow 3! arrangements of the three teenagers.

$\therefore$ Total no. of arrangements $= 7! \times 3! = 30240$

(ii) *Method 1*

One person is chairperson. One of the other nine, who is seated in one of the remaining nine chairs, is secretary.

$\therefore$ p(the secretary sits opposite the chairperson)$= \frac{1}{9}$.

Method 2

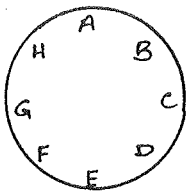
Seat the chairperson.

No. of arrangements for the other nine people$= 9!$

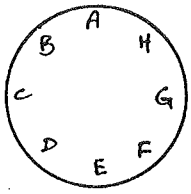
No. of arrangements if the secretary is sat opposite$= 8!$

$$\therefore \text{p(s. sits opposite)} = \frac{8!}{9!} = \frac{8!}{9 \times 8!} = \frac{1}{9}$$

Note: the results in part (i) could be 15120 if mirror image seating is disallowed, e.g.

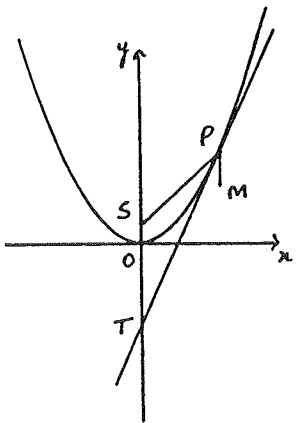


is counted as equivalent to



QUESTION 6

(a) (i)



$$x^2 = 4Ay$$

$$\therefore \frac{d}{dx}(x^2) = \frac{d}{dx}(4Ay)$$

$$2x = 4A.\frac{dy}{dx}$$

At the point $P(2Ap, Ap^2)$

$$2 \times 2Ap = 4A.\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = p$$

Tangent at P:

$$y - Ap^2 = p(x - 2Ap)$$

$$y - Ap^2 = px - 2Ap^2$$

$$px - y - Ap^2 = 0$$

(ii) Tangent meets $y-$axis at T when $x = 0$

$$\therefore y = -Ap^2$$

$$\therefore T(0, -Ap^2)$$

But $S(0,A)$

$$\therefore ST = A + Ap^2 = A(p^2 + 1)$$

$$\begin{aligned} SP^2 &= (2Ap - 0)^2 + (Ap^2 - A)^2 \\ &= 4A^2p^2 + A^2p^4 - 2A^2p^2 + A^2 \\ &= A^2p^4 + 2A^2p^2 + A^2 \\ &= A^2(p^4 + 2p^2 + 1) \\ &= A^2(p^2 + 1)^2 \end{aligned}$$

$$\therefore SP = A(p^2 + 1)$$

$$\therefore SP = ST$$

(iii)

$PM \parallel ST$

$\therefore \angle STP = \angle TPM$ (alternate angles)

$\angle STP = \angle SPT$ (base angles of isos. $\triangle SPT$)

$\therefore \angle TPM = \angle SPT$ (as required)

(b)

$$\begin{aligned} (7 + 3x)^{25} &= \Sigma_{k=0}^{25} t_k x^k \\ &= \Sigma_{k=0}^{25} \binom{25}{k} 7^{25-k}(3x)^k \\ &= \Sigma_{k=0}^{25} \binom{25}{k} 7^{25-k}.3^k.x^k \end{aligned}$$

(i) $\therefore t_k = \binom{25}{k} 7^{25-k}.3^k$

(ii)

$$\begin{aligned} \therefore \frac{t_{k+1}}{t_k} &= \frac{\binom{25}{k+1} 7^{25-(k+1)} 3^{k+1}}{\binom{25}{k} 7^{25-k} 3^k} \\ &= \frac{\frac{25!}{(k+1)!(24-k)!} \times 7^{24-k} \times 3^{k+1}}{\frac{25!}{k!(25-k)!} \times 7^{25-k} \times 3^k} \\ &= \frac{k!(25-k)!}{(k+1)!(24-k)!} \times \frac{7^{24-k}}{7^1 \times 7^{24-k}} \times \frac{3^1 \times 3^k}{3^k} \\ &= \frac{k! \times (25-k) \times (24-k)!}{(k+1) \times k! \times (24-k)!} \times \frac{1}{7} \times \frac{3}{1} \\ &= \frac{3(25-k)}{7(k+1)} \end{aligned}$$

(iii) Each coefficient is larger than the previous coefficient whilst ever $t_{k+1} > t_k$.

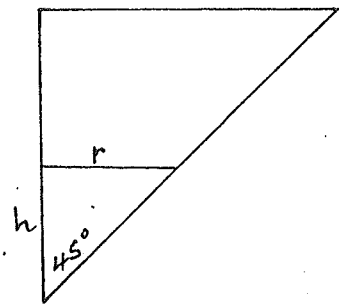
$$\text{i.e. if } \frac{t_{k+1}}{t_k} > 1$$
$$\text{i.e. when } \frac{3(25-k)}{7(k+1)} > 1$$
$$\text{i.e. } 75 - 3k > 7k + 7$$
$$68 > 10k$$
$$k < 6.8$$

i.e, $k = 6$ for which $\frac{t_7}{t_6} > 1$

i.e. $t_7 = \binom{25}{7} 7^{18} \times 3^7$ is the largest coefficient.

QUESTION 7

(a)



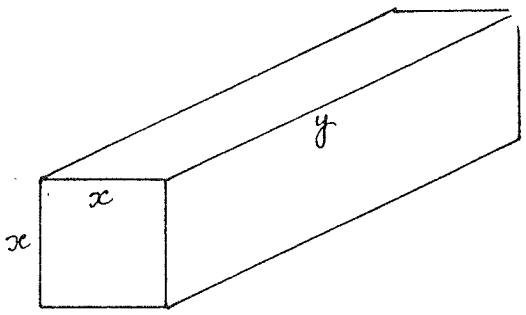
The semi-vertical angle = 45° and so $h = r$ always.

$$S = \pi r^2$$
$$\frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt}$$
$$= 2\pi r \frac{dr}{dt}$$
$$V = \tfrac{1}{3}\pi r^2 h = \tfrac{1}{3}\pi r^3$$
$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$
$$= \pi r^2 \frac{dr}{dt}$$
$$= \tfrac{1}{2} r (2\pi r \frac{dr}{dt})$$
$$\frac{dV}{dt} = \tfrac{1}{2} r \frac{dS}{dt}$$
$$\frac{dV}{dt} = 24\,\text{cm}^3/\text{sec}$$
$$r = 16\,\text{cm}$$
$$24 = \tfrac{1}{2} \times 16 \frac{dS}{dt}$$
$$\therefore \frac{dS}{dt} = 3$$

i.e. area is incresing at 3 cm^2/sec.

(b) $l + g \leq 100$

Case 1: $x < y$



i.e. $4x + y \leq 100$

For largest parcel:

$$4x + y = 100$$
$$\text{i.e. } y = 100 - 4x$$
$$V = x.x.y$$
$$= x^2(100 - 4x) \quad (0 < x < 25)$$
$$= 100x^2 - 4x^3$$
$$V' = 200x - 12x^2$$
$$V'' = 200 - 24x$$

For a st. pt. $V' = 0$

$$\text{i.e. } 4x(50 - 3x) = 0$$
$$x = \frac{50}{3}, 0$$

discard $x = 0$ since $x > 0$ for parcel to exist

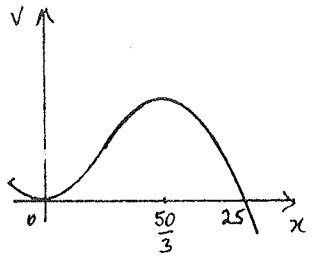
For $x = \frac{50}{3}$

$$V'' = 200 - 24 \times \tfrac{50}{3}$$
$$= 200 - 400 < 0$$

$\therefore$ Max. when $x = \frac{50}{3}$

$$\text{and } y = 100 - 4x$$
$$= 100 - 4 \times \tfrac{50}{3}$$
$$= 100(1 - \tfrac{2}{3})$$
$$= \frac{100}{3}$$

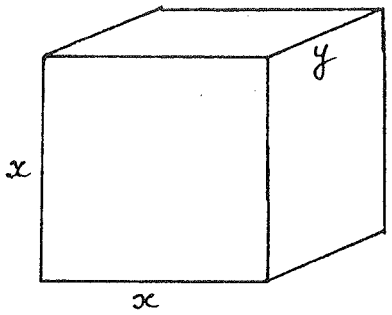
Note: V has a local maximum for $x = \frac{50}{3}$. It should also be shown that it is a global maximum for $0 < x < 25$. (next page)



From the sketch it is obviously true.
($V = 4x^2(25 - x)$ has a double root at 0, and one maximum for $0 < x < 25$.)

$$\begin{aligned}\text{Max. } V &= 4(\tfrac{50}{3})^2(25 - \tfrac{50}{3}) \\ &= \frac{4 \times 2500}{9} \times \frac{25}{3} \\ &= \frac{250000}{27} \\ &\approx 9259.3\,\text{cm}^3\end{aligned}$$

Case 2: $x > y$



$$\begin{aligned}\text{Girth} &= 2(x + y) \\ \therefore x + 2(x + y) &\leq 100 \\ 3x + 2y &\leq 100\end{aligned}$$

For largest parcel

$$\begin{aligned}y &= 50 - \frac{3x}{2} \\ V &= x^2 y \\ &= x^2\left(50 - \frac{3x}{2}\right) \\ &= 50x^2 - \tfrac{3}{2}x^3 \\ V' &= 100x - \tfrac{9}{2}x^2 \\ V'' &= 100 - 9x\end{aligned}$$

For a max. $V' = 0$, $V'' < 0$

$$V' = 0 \Rightarrow \quad x(100 - \tfrac{9}{2}x) = 0$$

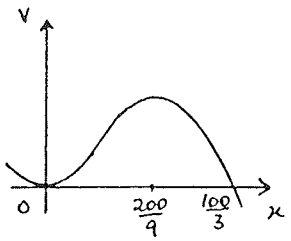
$$\therefore x = \frac{200}{9},\ 0 \text{ (discard } x = 0)$$

When $x = \dfrac{200}{9}$

$$V'' = 100 - 9 \times \tfrac{200}{9} < 0$$

$\therefore$ there is a max. when $x = \frac{200}{9}$

$$\begin{aligned}\text{and } y &= 50 - \tfrac{3}{2} \times \tfrac{200}{9} \\ &= 50 - \tfrac{100}{3} \\ &= \frac{50}{3}\end{aligned}$$

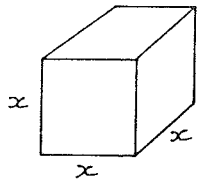


$$\begin{aligned}\text{Max } V &= x^2(50 - \tfrac{3}{2}x) \\ &= (\tfrac{200}{9})^2(50 - \tfrac{3}{2} \times \tfrac{200}{9} \\ &= \tfrac{40000}{81}(50 - \tfrac{100}{3}) \\ &= \tfrac{40000}{81} \times \tfrac{50}{3} \\ &\approx 8230.5\,\text{cm}^3\end{aligned}$$

Case 3: x = y

The parcel is a cube, with girth 4x, longest side x. This means that the condition $l + g \leqslant 100$ becomes $5x \leqslant 100$, i.e. $x \leqslant 20$.

The largest parcel under these conditions is a cube, side 20cm, volume 20^3 cm^3 = 8000 cm^3.



$\therefore$ Case 1 gives the maximum volume for dimensions

$$16\tfrac{2}{3}\,\text{cm} \times 16\tfrac{2}{3}\,\text{cm} \times 33\tfrac{1}{3}\,\text{cm}.$$
