HIGHER SCHOOL CERTIFICATE
1987 EXAMINATION PAPER
3 / 4 UNIT MATHEMATICS SOLUTIONS

QUESTION ONE

(i) (a) $y = \dfrac{1}{3+x^2} = \left(3+x^2\right)^{-1}$

Then $\dfrac{dy}{dx} = -1\left(3+x^2\right)^{-2}.2x$

$= \dfrac{-2x}{\left(3+x^2\right)^2}$

(b) $y = e^x \log_e(2x)$

Then $\dfrac{dy}{dx} = e^x \log_e(2x) + e^x.\dfrac{2}{2x}$

$= e^x\left(\log_e(2x) + \dfrac{1}{x}\right)$

(ii) (a) $\displaystyle\int (3x+2)^{10}\,dx = \frac{(3x+2)^{11}}{11\times 3} + c$

$= \frac{1}{33}(3x+2)^{11} + c$

(b) $\displaystyle\int \frac{5}{2+x^2}\,dx = \int \frac{5}{\left(\sqrt{2}\right)^2 + x^2}\,dx$

$= \dfrac{5}{\sqrt{2}}\tan^{-1}\left(\dfrac{x}{\sqrt{2}}\right) + c$

using the integral table.

(iii) (a) $y = x^4 + 4x^{3/2}$

$\Rightarrow \quad y' = 4x^3 + 6x^{1/2}$

$\Rightarrow$ At $(1,5)$ $m = 10$ and $m_\perp = -\frac{1}{10}$

So the normal at $(1,5)$ is

$y - 5 = -\frac{1}{10}(x-1)$

i.e. $x + 10y - 51 = 0$

(b) The gradient of $l : 2x + 3y - 7 = 0$ is $-\frac{2}{3}$

So the acute angle betwen l and the normal is θ where $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

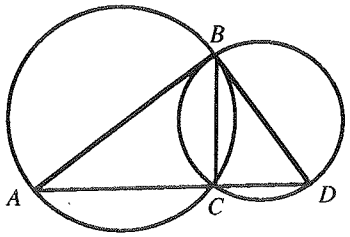
$= \left|\dfrac{-\frac{1}{10} - \left(-\frac{2}{3}\right)}{1 + \left(-\frac{1}{10}\right)\left(-\frac{2}{3}\right)}\right|$

$= \frac{17}{32}$

So $\theta \approx 28^0$

QUESTION TWO

(i) (a)



(b) Join AC, BC, DC.

Since AB is a diameter

$\angle ACB = 90^0$ (angle in a semicirle is a rt $\angle$)

Similarly $\angle BCD = 90^0$

$\therefore\ \angle ACB + \angle BCD = 180^0$

i.e. $\angle ACD$ is a straight angle

$\therefore$ A, C and D are collinear points.

(ii) $(x-2)^2 \geq 4 \quad \Rightarrow \quad x - 2 \leq -2$ or $x - 2 \geq 2$

$\Rightarrow \quad x \leq 0$ or $x \geq 4$

(iii) (a) $\dfrac{d}{dx}\left(x\sin^{-1}x + \left(1-x^2\right)^{1/2}\right)$

$= 1\sin^{-1}x + x.\dfrac{1}{\sqrt{1-x^2}} + \frac{1}{2}\left(1-x^2\right)^{-1/2}(-2x)$

$= \sin^{-1}x + \dfrac{x}{\sqrt{1-x^2}} - \dfrac{x}{\sqrt{1-x^2}}$

$= \sin^{-1}x$

(b) So $\displaystyle\int_0^1 \sin^{-1}x\,dx$

$= \left[x\sin^{-1}x + \sqrt{1-x^2}\right]_0^1$

$= \left(\sin^{-1}1 - 0\right) + (0-1)$

$= \frac{\pi}{2} - 1$

(iv) For $Y = \sin^{-1}X$: Domain is $-1 \leq X \leq 1$

& Range is $-\frac{\pi}{2} \leq Y \leq \frac{\pi}{2}$

So for $\dfrac{y}{2} = \sin^{-1}3x$:

Domain is $-1 \leq 3x \leq 1$: that is $-\frac{1}{3} \leq x \leq \frac{1}{3}$

& Range is $-\frac{\pi}{2} \leq \frac{y}{2} \leq \frac{\pi}{2}$: that is $-\pi \leq y \leq \pi$

QUESTION THREE

(i) $\sin 2x = \cos x$

$\Rightarrow \quad 2\sin x\cos x - \cos x = 0$

$\Rightarrow \quad \cos x(2\sin x - 1) = 0$

$\Rightarrow \quad \cos x = 0 \text{ or } \sin x = \frac{1}{2}$

$\Rightarrow \quad x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}.$

(ii) All the terms in the expansion of $\left(2x - \frac{1}{x}\right)^{11}$ have the form $^{11}C_r(2x)^{11-r}\left(-\frac{1}{x}\right)^r$

$= {}^{11}C_r 2^{11-r}(-1)^r x^{11-2r}$

For the term in x^3 : $11 - 2r = 3 \Rightarrow r = 4$

So the coefficient is $^{11}C_4 2^7(-1)^4 = 42240$

(iii) The probabilities come from a binomial probability distribution with $p = \frac{1}{5}$ (get black) and $q = \frac{4}{5}$ (don't get black.)

(a) $P(2 \text{ black jellybeans})$

$= {}^{10}C_2\left(\frac{1}{2}\right)^2\left(\frac{4}{5}\right)^8$

$= 0.302$ (to 3 decimal places)

(b) $P(\text{fewer than 2 blacks})$

$= P(0 \text{ blacks}) + P(1 \text{ black})$

$= {}^{10}C_0\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^{10} + {}^{10}C_1\left(\frac{1}{5}\right)^1\left(\frac{4}{5}\right)^9$

$= 0.376$ (to 3 decimal places)

(c) $P(0,1 \text{ or } 2 \text{ blacks})$

$= 0.302 + 0.376$ (from above)

$= 0.678$

$P(\text{more than } 2)$

$= 1 - P(0,1 \text{ or } 2 \text{ blacks})$

$= 0.322$

So $P(\text{more than } 2) < P(\text{fewer than } 2)$

So it is more likely that the sample has fewer than 2 black jellybeans.

QUESTION FOUR

(i) $V = \pi\int_0^2 y^2\,dx$

$= \pi\int_0^2 x^2\left(8 - x^3\right)^8\,dx$

Putting $u = 8 - x^3$

$\Rightarrow \quad du = -3x^2dx \text{ so } dx = -\frac{du}{3x^2}$

and $x = 0 \Rightarrow u = 8$ and $x = 2 \Rightarrow u = 0$

$\therefore \quad V = \pi\int_8^0 \left(-\frac{1}{3}u^8\right)du$

$= \frac{\pi}{3}\left[-\frac{u^9}{9}\right]_8^0$

$= \frac{\pi}{3}\left[0 + \frac{8^9}{9}\right]$

$= \frac{2^{27}\pi}{27} \text{ units}^3$

(ii) (a) Consider a position of the pebble during the flight. The only force acting is gravity g downwards.

Horizontal motion :

$\ddot{x} = 0$

$\therefore \quad \dot{x} = c_1$

When $t = 0,\ \dot{x} = 20\cos 30^0$

$= 10\sqrt{3} \quad \therefore \quad c_1 = 10\sqrt{3}$

$\therefore \quad \dot{x} = 10\sqrt{3}$

$\therefore \quad x = 10\sqrt{3}\,t + c_2$

When $t = 0,\ x = 0 \quad \therefore \quad c_2 = 0$

$\therefore \quad x = 10\sqrt{3}\,t$

Vertical motion :

$\ddot{y} = -g$

$\therefore \quad \dot{y} = -gt + c_3$

When $t = 0,\ \dot{y} = 20\sin 30^0$

$= 10 \quad \therefore \quad c_3 = 10$

$\therefore \quad \dot{y} = -gt + 10$

$\therefore \quad y = -\frac{1}{2}gt^2 + 10t + c_4$

When $t = 0,\ y = 40 \quad \therefore \quad c_4 = 40$

$\therefore \quad y = -\frac{1}{2}gt^2 + 10t + 40$

$\therefore \quad x(t) = 10\sqrt{3}\,t$

& $y(t) = -\frac{1}{2}gt^2 + 10t + 40$

(b) Taking $g = 10$:

When the pebble hits the lake : $y = 0$:

$\Rightarrow \quad -5t^2 + 10t + 40 = 0$

$\Rightarrow \quad t^2 - 2t - 8 = 0$

$\Rightarrow \quad (t - 4)(t + 2) = 0$

$\Rightarrow \quad t = -2, 4$

Using $t = 4$: $x = 40\sqrt{3}$

$\therefore$ After 4 seconds, the pebble hits the lake at a distance of $40\sqrt{3}$ metres from the foot of the cliff.

QUESTION FIVE

(a) Since there is a root at $x=-2$, $(x+2)$ is a factor.
Dividing to find the other root :

$$\begin{array}{r} 8x^2-4x-10 \\ x+2\overline{)\,8x^3+12x^2-18x-20} \\ \underline{8x^3+16x^2} \\ -4x^2-18x-20 \\ \underline{-4x^2-8x} \\ -10x-20 \\ \underline{-10x-20} \end{array}$$

So $f(x)=(x+2)\left(8x^2-4x-10\right)$

$\Rightarrow \quad x=-2 \quad$ or $\quad 8x^2-4x-10=0$

$$x=\frac{2\pm\sqrt{84}}{8}$$

So the three roots are $2,\ \dfrac{1\pm\sqrt{21}}{4}$

Alternative method :
$(x+2)$ is a factor and the quadratic factor is of the form : $\left(8x^2+kx-10\right)$

i.e. $8x^3+12x^2-18x-20\equiv(x+2)\left(8x^2+kx-10\right)$

When $x=1$: $\quad -18=3(k-2)$

$\Rightarrow \quad k=-4$

So $f(x)=(x+2)\left(8x^2-4x-10\right)$

and the three roots are $2,\ \dfrac{1\pm\sqrt{21}}{4}$

(b) $f(x)=8x^3+12x^2-18x-20$

$f'(x)=24x^2+24x-18$

For the stationary points : $f'(x)=0$

$\Rightarrow \quad 24x^2+24x-18=0$

$\Rightarrow \quad 6(2x+3)(2x-1)=0$

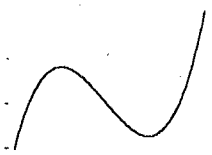
$\Rightarrow \quad x=-1\frac{1}{2},\ \frac{1}{2}.$

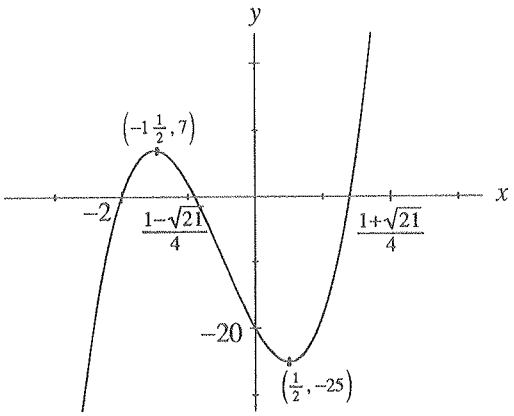
$f\left(-1\frac{1}{2}\right)=7$ and $f\left(\frac{1}{2}\right)=-25$

$\therefore$ There are stationary points at $\left(-1\frac{1}{2},7\right)$ & $\left(\frac{1}{2},-25\right)$

$f(0)=-20$ so the graph cuts the y axis at $(0,-20)$

Since the leading coefficient is positive, the graph will have the shape :





(c) By Newton's method, if x_1 is a good first approximation to a root, then a better approximation is given by :

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}$$

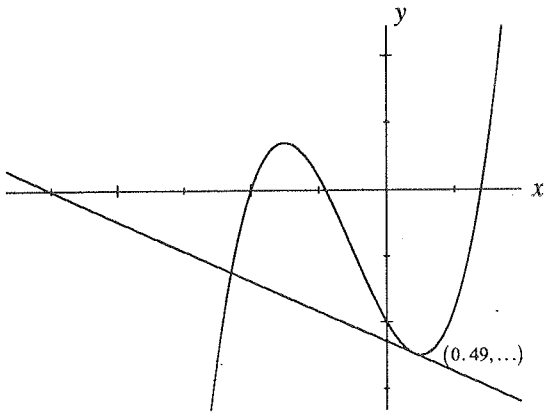
$f(x)=8x^3+12x^2-18x-20$

So $\quad f'(x)=24x^2+24x-18$

Using $x_1=1$: $\quad x_2=1-\dfrac{-18}{30}=1.6$

So a better approximation is 1.6

(d) Newton's method draws the tangent at the point $\left(x_1,f(x_1)\right)$ and makes x_2 equal to the value of the x intercept of this tangent.
Diagramatically using $x_1=0.49$:



From the diagram we can see that this series of approximations will approach the root -2.

QUESTION SIX

(i) (a) If $T = S + Be^{kt}$

$$\frac{dT}{dt} = Bke^{kt}$$

$$= k(T - S) \quad \text{since } T - S = Be^{kt}$$

$\Rightarrow$ $T = S + Be^{kt}$ is a solution of $\frac{dT}{dt} = k(T - S)$

(b) Given : $S = 20$,

$t = 0$ when $T = 80$,

and $t = 2$ when $T = 40$.

$t = 0, T = 80 \Rightarrow 80 = 20 + Be^0$

$\Rightarrow B = 60$

$t = 2, T = 40 \Rightarrow 40 = 20 + 60e^{2k}$

$\Rightarrow e^{2k} = \frac{1}{3}$

$\Rightarrow k = \frac{1}{2}\ln\left(\frac{1}{3}\right) \approx -0.5493$

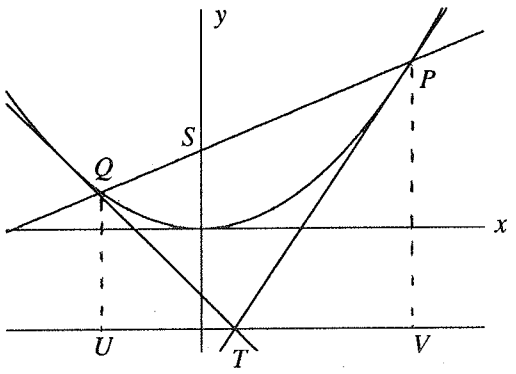
$\Rightarrow T = 20 + 60e^{-0.5493t}$

When $t = 3$: $T = 20 + 60e^{-1.6479} \approx 31.547$

The temperature of the body will be 32^0

(to the nearest degree)

(ii) (a)



For $P(2Ap, Ap^2)$ & $Q(2Aq, Aq^2)$

$$m_{PQ} = \frac{Ap^2 - Aq^2}{2Ap - 2Aq}$$

$$= \frac{A(p+q)(p-q)}{2A(p-q)}$$

$$= \tfrac{1}{2}(p+q)$$

Equation of PQ:

$$y - Ap^2 = \tfrac{1}{2}(p+q)(x - 2Ap)$$

If this passes through $(0, A)$

$$A - Ap^2 = \tfrac{1}{2}(p+q)(-2Ap)$$

$\Rightarrow$ $1 - p^2 = -p^2 - pq$

$\Rightarrow$ $pq = -1$

(b) Tangent at P: $y = px - Ap^2$

Tangent at Q: $y = qx - Aq^2$

Solving these equations :

$$0 = (p-q)x - A(p^2 - q^2)$$

$\Rightarrow$ $(p-q)x = A(p-q)(p+q)$

$\Rightarrow$ $x = A(p+q)$

& $y = pA(p+q) - Ap^2$

$$= Ap^2 + Apq - Ap^2$$

$$= Apq$$

Since $pq = -1$, $y = -A$.

So the tangents intersect on the line $y = -A$

(c) $PQ^2 = \left(2Ap - 2A.\frac{-1}{p}\right)^2 + \left(Ap^2 - A.\frac{1}{p^2}\right)^2$

$$= A^2\left(4p^2 + 8 + \frac{4}{p^2} + p^4 - 2 + \frac{1}{p^4}\right)$$

$$= A^2\left(p^4 + 4p^2 + 6 + \frac{4}{p^2} + p^4\right)$$

$$= A^2\left(p + \frac{1}{p}\right)^4$$

Using $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

So $PQ = A\left(p + \frac{1}{p}\right)^2$

Alternative :

If U and V are points on the directrix $y = -A$ at the foot of the perpendiculars from Q and P, then using the property of parabolas :

$$PS = PV = AP^2 + A$$

and $QS = QU = Aq^2 + A$

So $PQ = PS + QS$

$$= Ap^2 + A + Aq^2 + A$$

$$= A(p^2 + 2 + q^2)$$

$$= A(p^2 - 2pq + q^2) \quad \because pq = -1$$

$$= A(p-q)^2$$

$$= A\left(p + \frac{1}{p}\right)^2 \quad \because q = -\frac{1}{p}$$