2012 NSW BOS Mathematics Exam Solutions

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Section I

1	2	3	4	5	6	7	8	9	10
B	D	C	A	C	D	C	A	B	B

Q1 B

Q2 $\dfrac{1}{2\sqrt{5}-\sqrt{3}}=\dfrac{1}{2\sqrt{5}-\sqrt{3}}\times\dfrac{2\sqrt{5}+\sqrt{3}}{2\sqrt{5}+\sqrt{3}}=\dfrac{2\sqrt{5}+\sqrt{3}}{17}$ D

Q3 $x^2+3x-1=0$ has roots α and β,
$\therefore\ \alpha\beta=-1,\ \alpha+\beta=-3$
$\therefore\ \alpha\beta+(\alpha+\beta)=-1-3=-4$ C

Q4 At $x=a$, the gradient of the curve is *positive*, and it is *decreasing* as x increases, $\therefore\ f'(a)>0$ and $f''(a)<0$ A

Q5 Let (x,y) be a point on the line $y=3x+1$,
$\therefore\ (x,y)$ is $(x,3x+1)$
Distance ℓ *between* (x,y) *and* $(2,-1)$
$=\sqrt{(x-2)^2+(3x+1-(-1))^2}=\sqrt{10x^2+8x+8}$
Let $\dfrac{d\ell}{dx}=\dfrac{20x+8}{2\sqrt{10x^2+8x+8}}=0$, $\therefore\ x=-\dfrac{2}{5}$
$\therefore$ *perpendicular distance* $=\sqrt{10\left(-\dfrac{2}{5}\right)^2+8\left(-\dfrac{2}{5}\right)+8}=\dfrac{8}{\sqrt{10}}$ C

Alternative method: (1) Find the equation of the line perpendicular to $y=3x+1$. (2) Find the point of intersection of the two lines. (3) Find the distance between $(2,-1)$ and the point of intersection.

Q6 $\sqrt{3}\tan x=-1,\ 0\le x\le 2\pi$
$\tan x=-\dfrac{1}{\sqrt{3}},\ x=\dfrac{5\pi}{6},\ \dfrac{11\pi}{6}$ D

Q7 $a=e^x,\ a^2=\left(e^x\right)^2=e^{2x},\ \log_e\left(a^2\right)=2x$ C

Q8 Pick a point in the shaded region, say $(3,0)$, to test the inequalities. A

Q9 $\int_1^4\dfrac{1}{3x}dx=\dfrac{1}{3}\int_1^4\dfrac{1}{x}dx=\dfrac{1}{3}\left[\ln x\right]_1^4=\dfrac{1}{3}\ln 4$ B

Q10 The values of $\int_2^7 f(x)dx$ and $\int_2^8 f(x)dx$ are negative.
$\therefore\ \int_0^2 f(x)dx$ has the greatest value. B

Section II

Q11a $2x^2-7x+3=(2x-1)(x-3)$

Q11b $|3x-1|<2$
$(3x-1)^2<4,\ 9x^2-6x+1<4$
$\therefore\ 9x^2-6x-3<0,\ 3x^2-2x-1<0,\ (3x+1)(x-1)<0$
$\left(x+\dfrac{1}{3}\right)(x-1)<0$, $\therefore\ -\dfrac{1}{3}<x<1$

Alternative method:
$3x-1<2$ when $3x-1\ge 0$ and $-(3x-1)<2$ when $3x-1<0$
$\therefore\ x<1$ when $x\ge\dfrac{1}{3}$ and $x>-\dfrac{1}{3}$ when $x<\dfrac{1}{3}$
$\therefore\ -\dfrac{1}{3}<x<1$

Q11c At $x=3,\ y=x^2=9,\ \dfrac{dy}{dx}=2x=6$
Equation of the tangent: $y-y_1=m(x-x_1)$
$y-9=6(x-3)$, $\therefore\ y=6x-9$

Q11d $\dfrac{d}{dx}\left(3+e^{2x}\right)^5=5\left(3+e^{2x}\right)^4\left(2e^{2x}\right)=10e^{2x}\left(3+e^{2x}\right)^4$

Q11e $x^2=16(y-2)=4\times 4(y-2)$
The y-coordinate of the focus is given by $y-2=4$, i.e. $y=6$.
The point $(0,6)$ is the focus.

Q11f $A=\dfrac{1}{2}r^2\theta$, $\therefore\ A=\dfrac{1}{2}r\ell$ where $\ell=r\theta$ is the length of the arc, $\therefore\ 50=\dfrac{1}{2}\times 6\times\ell$, $\ell=\dfrac{50}{3}$ cm

Q11g $\int_0^{\frac{\pi}{2}}\sec^2\dfrac{x}{2}dx=2\left[\tan\dfrac{x}{2}\right]_0^{\frac{\pi}{2}}=2\tan\dfrac{\pi}{4}=2$

Q12ai $\dfrac{d}{dx}\left((x-1)\log_e x\right)=\left(\dfrac{d}{dx}(x-1)\right)(\log_e x)+(x-1)\dfrac{d}{dx}\log_e x$
$=\log_e x+(x-1)\times\dfrac{1}{x}=\log_e x+\dfrac{x-1}{x}$

Q12aii $\dfrac{d}{dx}\left(\dfrac{\cos x}{x^2}\right)=\dfrac{\left(x^2\right)\left(\dfrac{d}{dx}\cos x\right)-(\cos x)\left(\dfrac{d}{dx}x^2\right)}{x^4}$
$=\dfrac{x^2(-\sin x)-2x\cos x}{x^4}=-\dfrac{x(x\sin x+2\cos x)}{x^4}=-\dfrac{x\sin x+2\cos x}{x^3}$

Q12b $\int\dfrac{4x}{x^2+6}dx=2\int\dfrac{2x}{x^2+6}dx=2\int\dfrac{1}{u}du$
$=2\ln u+c=2\ln\left(x^2+6\right)+c$

Let $u=x^2+6$
$\dfrac{du}{dx}=2x$

Q12ci Arithmetic sequence: $a=3$, $d=2$

$t_{20}=3+(20-1)\times 2=41$

Q12cii $S_{20}=\frac{20}{2}(3+41)=440$

Q12ciii $S_n=\frac{n}{2}(2\times 3+(n-1)\times 2)\le 200$

$n(6+2n-2)\le 400$, $2n^2+4n\le 400$, $n^2+2n+1\le 201$

$(n+1)^2\le 201$, $n+1\le\sqrt{201}$, $n\le\sqrt{201}-1$

$\therefore\ n=13$

Q12di $Area\approx 3(0.5+4\times 2.3+2\times 2.9+4\times 3.8+2.1)=98.4\text{ m}^2$

Q12dii $Volume\approx 98.4\times 0.4\times 10=393.6\text{ m}^3$

Q13ai $A(4,0)$, $B(0,8)$, $\therefore\ AB=\sqrt{(0-4)^2+(8-0)^2}=4\sqrt{5}$

Q13aii The cosine rule: $b^2=c^2+a^2-2ca\cos\angle ABC$

$5^2=\left(4\sqrt{5}\right)^2+\left(\sqrt{65}\right)^2-2\left(4\sqrt{5}\right)\left(\sqrt{65}\right)\cos\angle ABC$

$25=80+65-8\sqrt{325}\cos\angle ABC$, $\cos\angle ABC=\frac{80+65-25}{8\sqrt{325}}$,

$\therefore\ \angle ABC\approx 34°$

Q13aiii Let $N(p,q)$. $\therefore\ 2p+q=8$ …… (1)

$m_{CN}\times m_{AB}=-1$, $\therefore\ \frac{q-4}{p-7}\times -2=-1$ …… (2)

From (1), $q=8-2p$ …… (3)

Substitute (3) in (2), $\frac{8-2p-4}{p-7}\times 2=1$, $\therefore\ 8-4p=p-7$,

$\therefore\ p=3$ and $q=2$, $\therefore\ N(3,2)$

Q13bi Solve $y=x^2-3x$ and $y=5x-x^2$ to find x, $x>0$.

$x^2-3x=5x-x^2$, $\therefore\ 2x^2-8x=0$, $2x(x-4)=0$

$\therefore\ x=4$ is the x-coordinate of point A.

Q13bii *Area of the shaded region* $=\int_0^4\left[\left(5x-x^2\right)-\left(x^2-3x\right)\right]dx$

$=\int_0^4\left(8x-2x^2\right)dx=\left[4x^2-\frac{2x^3}{3}\right]_0^4=\frac{64}{3}$

Q13ci $\Pr(RR)=\frac{3}{5}\times\frac{3}{7}=\frac{9}{35}$

Q13cii $\Pr(at.least.one.white)=1-\Pr(RR)=1-\frac{9}{35}=\frac{26}{35}$

Q13ciii $\Pr(same.colour)=\Pr(RR)+\Pr(WW)=\frac{9}{35}+\frac{2}{5}\times\frac{4}{7}=\frac{17}{35}$

Q14ai $f(x)=3x^4+4x^3-12x^2$

$f'(x)=12x^3+12x^2-24x=12x\left(x^2+x-2\right)=12x(x-1)(x+2)$

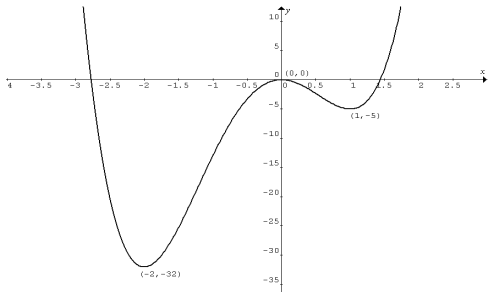
Let $f'(x)=0$, $12x(x-1)(x+2)=0$

$\therefore\ x=-2$, 0, 1 and the corresponding $y=-32$, 0, -5.

The stationary points are $(-2,-32)$, $(0,0)$ and $(1,-5)$.

x		-2		0		1	
$f'(x)$	–	0	+	0	–	0	+
Nature		local min		local max		local min	

Q14aii



Q14aiii The function is increasing in the intervals $-2<x<0$ and $x>1$.

Q14aiv $3x^4+4x^3-12x^2+k=0$ will have no solution if $k>32$.

Q14b $Volume=\int_0^1\pi\left(\frac{3}{(x+2)^2}\right)^2dx=\int_0^1\pi\frac{9}{(x+2)^4}dx$

$=9\pi\int_0^1(x+2)^{-4}dx=9\pi\left[-\frac{1}{3(x+2)^3}\right]_0^1=9\pi\left(-\frac{1}{81}+\frac{1}{24}\right)$

$=\frac{19\pi}{72}$.

Q14ci $N(t)=1000e^{kt}$, when $t=20$, $N=2000$

$\therefore\ 1000e^{20k}=2000$, $e^{20k}=2$, $k=\frac{\ln 2}{20}\approx 0.0347$

Q14cii $N(120)=1000e^{120k}=1000\left(e^{20k}\right)^6=1000\times 2^6=64000$

Q14ciii $\frac{dN}{dt}=kN(t)$

When $t=120$, $\frac{dN}{dt}=kN(120)=\frac{\ln 2}{20}\times 64000=3200\ln 2$

≈ 2218 per minute

Q14civ When $N=100000$, $1000\left(e^{20k}\right)^{\frac{t}{20}}=100000$,

$\therefore\ 2^{\frac{t}{20}}=100$, $\frac{t}{20}=\frac{\log_{10}100}{\log_{10}2}$, $t=\frac{40}{\log_{10}2}\approx 133$ minutes

Q15ai Geometric series: $a = 10$, $r = 96\% = 0.96$

$S_{10} = \dfrac{10\left(1 - 0.96^{10}\right)}{1 - 0.96} \approx 83.8$ cm

The length required is 83.8 cm approximately.

Q15aii $S_\infty = \dfrac{10}{1 - 0.96} = 250$ cm or 2.5 m

∴ a 3-m strip is sufficient to make any number of rectangles.

Q15bi Assume that 'displacement' means displacement from the origin or simply position (relative to the origin).

$\dot{x} = 1 - 2\cos t$, at $t = 0$, $\dot{x} = 1 - 2\cos 0 = -1$

The initial velocity is 1 m s^{-1} to the left towards the origin.

Q15bii The velocity is maximum when $\cos t = -1$, $\dot{x} = 3$ m s^{-1}.

Q15biii $x = \int (1 - 2\cos t)dt = t - 2\sin t + c$

At $t = 0$, $x = 3$, ∴ $c = 3$.

∴ $x = t - 2\sin t + 3$

Q15biv When $\dot{x} = 1 - 2\cos t = 0$, $\cos t = \dfrac{1}{2}$, $t = \dfrac{\pi}{3}$.

$x = \dfrac{\pi}{3} - 2\sin\dfrac{\pi}{3} + 3 = \dfrac{\pi}{3} - \sqrt{3} + 3 \approx 2.32$

Q15ci

$\$A_1 = \$360000\left(1 + \dfrac{6}{12 \times 100}\right) - \$M = \$(360000 \times 1.005 - M)$

∴ $\$A_2 = \$(360000 \times 1.005 - M) \times 1.005 - \$M$

$= \$\left(360000 \times 1.005^2 - 1.005M - M\right)$

or $\$A_2 = \$\left(360000 \times 1.005^2 - \dfrac{M\left(1.005^2 - 1\right)}{1.005 - 1}\right)$ by formula.

Q15cii $A_{300} = 360000 \times 1.005^{300} - \dfrac{M\left(1.005^{300} - 1\right)}{1.005 - 1} = 0$

$M = \dfrac{360000 \times 1.005^{300} \times 0.005}{1.005^{300} - 1} \approx 2319.485 \approx 2319.50$ dollars

Q15ciii $\$A_n < \$180000$,

$360000 \times 1.005^n - \dfrac{2319.485\left(1.005^n - 1\right)}{1.005 - 1} < 180000$

$360000 \times 1.005^n - 463897.009\left(1.005^n - 1\right) < 180000$

$103897 \times 1.005^n > 283897$, $1.005^n > 2.732485$

$n > \dfrac{\ln 2.732485}{\ln 1.005} = 201.54$, ∴ after 202 months.

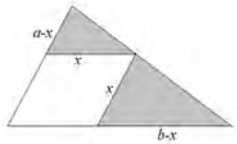
Q16ai $\angle EBF = \angle AED$, corresponding angles are equal

$\angle BEF = \angle EAD$, corresponding angles are equal

∴ $\angle BFE = \angle EDA$

Hence ΔEBF and ΔAED are similar.

Q16aii



$\dfrac{a - x}{x} = \dfrac{x}{b - x}$, ∴ $x^2 = (a - x)(b - x)$, $x^2 = ab - (a + b)x + x^2$

$(a + b)x = ab$, ∴ $x = \dfrac{ab}{a + b}$

Q16bi $m_{OT} = \tan\theta = \dfrac{\sin\theta}{\cos\theta}$, ∴ $m_{PT} = -\dfrac{1}{m_{OT}} = -\dfrac{\cos\theta}{\sin\theta}$

On the unit circle, point $T(\cos\theta, \sin\theta)$.

Equation of tangent *PT*:

$y - \sin\theta = -\dfrac{\cos\theta}{\sin\theta}(x - \cos\theta)$, $x\cos\theta + y\sin\theta = \sin^2\theta + \cos^2\theta$

∴ $x\cos\theta + y\sin\theta = 1$

Q16bii $x\cos\theta + y\sin\theta = 1$

When $y = 1$, $x = \dfrac{1 - \sin\theta}{\cos\theta}$, ∴ $Q\left(\dfrac{1 - \sin\theta}{\cos\theta}, 1\right)$, ∴ $BQ = \dfrac{1 - \sin\theta}{\cos\theta}$.

Q16biii $x\cos\theta + y\sin\theta = 1$

When $y = 0$, $x = \dfrac{1}{\cos\theta}$, ∴ $P\left(\dfrac{1}{\cos\theta}, 0\right)$, ∴ $OP = \dfrac{1}{\cos\theta}$.

Area of trapezium $A = \dfrac{1}{2}\left(\dfrac{1 - \sin\theta}{\cos\theta} + \dfrac{1}{\cos\theta}\right) \times 1 = \dfrac{2 - \sin\theta}{2\cos\theta}$.

Q16biv $\dfrac{dA}{d\theta} = \dfrac{(2\cos\theta)(-\cos\theta) - (2 - \sin\theta)(-2\sin\theta)}{4\cos^2\theta}$

$= \dfrac{2\sin\theta - 1}{2\cos^2\theta}$. For minimum area, let $\dfrac{dA}{d\theta} = 0$, $\dfrac{2\sin\theta - 1}{2\cos^2\theta} = 0$,

∴ $2\sin\theta - 1 = 0$, $\sin\theta = \dfrac{1}{2}$, ∴ $\theta = \dfrac{\pi}{6}$.

Q16ci Solve $x^2 + (y - c)^2 = r^2$ and $y = x^2$ simultaneously to find the *y*-coordinate of the points of touching.

By substitution, $y + (y - c)^2 = r^2$

Expand and simplify, $y^2 + (1 - 2c)y + \left(c^2 - r^2\right) = 0$

Solve for *y* using the quadratic formula and simplify,

$y = \dfrac{2c - 1 \pm \sqrt{1 - 4c + 4r^2}}{2}$.

Since the two points have the same *y*-coordinate,

∴ $1 - 4c + 4r^2 = 0$, ∴ $4c = 1 + 4r^2$

Q16cii From part ci, and $y > 0$, ∴ $y = \dfrac{2c - 1}{2} > 0$, ∴ $c > \dfrac{1}{2}$.

Please inform mathline@itute.com re conceptual and/or mathematical errors.