2010 NSW BOS Mathematics Exam Solutions

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Q1a $x^2 = 4x$, $x^2 - 4x = 0$, $x(x-4) = 0$, $x = 0$ or 4

Q1b $a + b\sqrt{5} = \dfrac{1}{\sqrt{5}-2} \times \dfrac{\sqrt{5}+2}{\sqrt{5}+2} = 2 + \sqrt{5}$, $\therefore\ a = 2$, $b = 1$

Q1c $(x+1)^2 + (y-2)^2 = 25$

Q1d $|2x+3| = 9$, $2x+3 = -9$ or $2x+3 = 9$, $\therefore\ x = -6$ or 3

Q1e By the product rule:

$$\frac{d}{dx}\left(x^2 \tan x\right) = \left(\frac{d}{dx}x^2\right)(\tan x) + x^2 \frac{d}{dx}\tan x = 2x \tan x + x^2 \sec^2 x$$

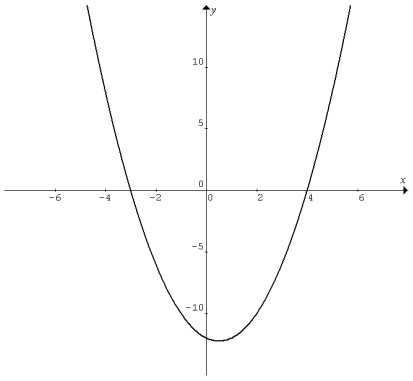
Q1f Geometric series: $a = 1$, $r = -\dfrac{1}{3}$, $S_\infty = \dfrac{a}{1-r} = \dfrac{1}{\frac{4}{3}} = \dfrac{3}{4}$

Q1g $f(x) = \sqrt{x-8}$, $x - 8 \geq 0$, $x \geq 8$, dom $f(x)$ is $[8, \infty)$

Q2a By the quotient rule:

$$\frac{d}{dx}\left(\frac{\cos x}{x}\right) = \frac{x(-\sin x) - \cos x}{x^2} = \frac{-x\sin x - \cos x}{x^2}$$

Q2b $x^2 - x - 12 < 0$, $(x+3)(x-4) < 0$, $-3 < x < 4$



Q2c $y = \ln(3x)$, $m = \dfrac{dy}{dx} = \dfrac{1}{x}$. At $x = 2$, $m = \dfrac{1}{x} = \dfrac{1}{2}$

Q2di $\displaystyle\int \sqrt{5x+1}\,dx = \int (5x+1)^{\frac{1}{2}}\,dx = \frac{2(5x+1)^{\frac{3}{2}}}{15} + c$

Q2dii Let $u = 4 + x^2$, $\dfrac{1}{2}\dfrac{du}{dx} = x$

$$\int \frac{x}{4+x^2}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln\left(4+x^2\right) + c$$

Q2e $\displaystyle\int_0^6 (x+k)\,dx = 30$, $\left[\dfrac{(x+k)^2}{2}\right]_0^6 = 30$, $\dfrac{(6+k)^2}{2} - \dfrac{k^2}{2} = 30$,

$(6+k)^2 - k^2 = 60$, $6(6+2k) = 60$, $k = 2$

Q3ai Midpoint $M\left(\dfrac{-2+12}{2}, \dfrac{-4+6}{2}\right) = M(5,1)$

Q3aii Gradient of $BC = \dfrac{8-6}{6-12} = -\dfrac{1}{3}$

Q3aiii Gradient of $MN = \dfrac{2-1}{2-5} = -\dfrac{1}{3}$

$\therefore$ MN and BC are parallel

$\therefore$ corresponding angles $\angle AMN = \angle ABC$ and corresponding angles $\angle ANM = \angle ACB$

Hence $\Delta AMN, \Delta ABC$ are similar.

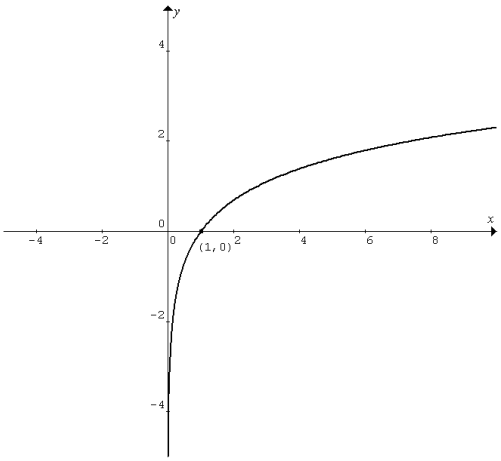
Q3aiv Equation of MN: $y - 2 = -\dfrac{1}{3}(x-2)$, $y = -\dfrac{1}{3}x + \dfrac{8}{3}$ or $x + 3y = 8$

Q3av Length of $BC = \sqrt{(12-6)^2 + (6-8)^2} = \sqrt{40} = 2\sqrt{10}$

Q3avi Let h be the perpendicular distance from A to BC.

$\dfrac{1}{2} \times 2\sqrt{10} \times h = 44$, $h = \dfrac{44}{\sqrt{10}} = \dfrac{22\sqrt{10}}{5}$

Q3bi $y = \ln x$

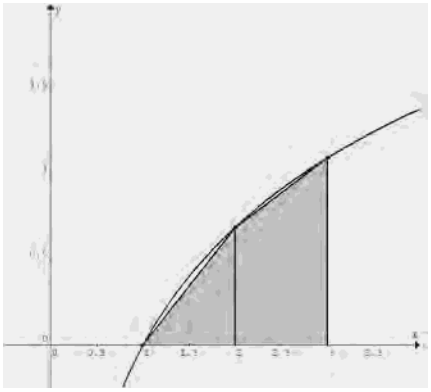


Q3bii $x = 1$, $y = 0$; $x = 2$, $y = \ln 2$; $x = 3$, $y = \ln 3$

$$\int_1^3 \ln x\,dx \approx \frac{1}{2}(0 + 2\ln 2 + \ln 3)(1) = \ln 2 + \frac{1}{2}\ln 3$$

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Q3biii The approximation (total area of the two trapeziums) is less than the exact value of $\int_1^3 \ln x\,dx$ (the area under $y=\ln x$ between $x=1$ and $x=3$).



Q4ai Arithmetic sequence:
$a=1,\ d=0.75,\ t_9=1+8\times0.75=7$
∴ 7 km in the 9th week

Q4aii $t_n=1+(n-1)0.75=10,\ n=13$, ∴ the 13th week

Q4aiii $S_{13}=\frac{13}{2}\left(2(1)+(13-1)0.75\right)=71.5$
Total distance $=71.5+13\times10=201.5$ km

Q4b Area of the enclosed region $=\int_0^2\left(e^{2x}-e^{-x}\right)dx$

$$=\left[\frac{e^{2x}}{2}-\frac{e^{-x}}{-1}\right]_0^2=\left(\frac{e^4}{2}+e^{-2}\right)-\left(\frac{1}{2}+1\right)=\frac{e^4}{2}+e^{-2}-\frac{3}{2}$$

Q4ci $\Pr(both.m.centres)=\frac{{}^4C_2}{{}^{12}C_2}=\frac{6}{66}=\frac{1}{11}$

Q4cii $\Pr(same.centre)=3\times\frac{1}{11}=\frac{3}{11}$

Q4ciii $\Pr(different.centres)=1-\frac{3}{11}=\frac{8}{11}$

Q4d $f(x)=1+e^x$
$f(x)\times f(-x)=\left(1+e^x\right)\left(1+e^{-x}\right)=1+e^x+e^{-x}+1=f(x)+f(-x)$

Q5ai $V=\pi r^2h=10$, ∴ $h=\frac{10}{\pi r^2}$
∴ $A=2\pi r^2+2\pi rh=2\pi r^2+2\pi r\left(\frac{10}{\pi r^2}\right)=2\pi r^2+\frac{20}{r}$

Q5aii As $r\to0^+,\ A\to\infty$; as $r\to\infty,\ A\to\infty$.
Also $A=2\pi r^2+\frac{20}{r}$ is a continuous function of r for $r>0$.
∴ A has a minimum value.
Let $\frac{dA}{dr}=4\pi r-\frac{20}{r^2}=0$, ∴ $r^3=\frac{5}{\pi}$, $r=\sqrt[3]{\frac{5}{\pi}}\approx1.17$ m
The minimum value occurs when $r=\sqrt[3]{\frac{5}{\pi}}\approx1.17$ m.

Q5bi $RHS=\frac{1+\sin x}{\cos^2x}=\frac{1}{\cos^2x}+\frac{\sin x}{\cos^2x}$
$=\left(\frac{1}{\cos x}\right)^2+\left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right)=\sec^2x+\sec x\tan x=LHS$

Q5bii $\sec^2x+\sec x\tan x$
$=\frac{1+\sin x}{\cos^2x}=\frac{1+\sin x}{1-\sin^2x}=\frac{1+\sin x}{(1-\sin x)(1+\sin x)}=\frac{1}{1-\sin x}$

Q5biii $\int_0^{\frac{\pi}{4}}\frac{1}{1-\sin x}dx=\int_0^{\frac{\pi}{4}}\left(\sec^2x+\sec x\tan x\right)dx$
$\left[\tan x+\sec x\right]_0^{\frac{\pi}{4}}=\left(\tan\frac{\pi}{4}+\sec\frac{\pi}{4}\right)-(\tan0+\sec0)$
$=1+\sqrt{2}-1=\sqrt{2}$

Q5c $\int_a^1\frac{1}{x}dx=\left[\ln x\right]_a^1=\ln1-\ln a=-\ln a=1,\ a=e^{-1}$
$\int_1^b\frac{1}{x}dx=\left[\ln x\right]_1^b=\ln b-\ln1=\ln b=1,\ b=e$

Q6ai $f(x)=(x+2)\left(x^2+4\right)$,
$f'(x)=(x+2)(2x)+\left(x^2+4\right)=3x^2+4x+4\neq0$ because
$\Delta=4^2-4(3)(4)<0$
∴ $f(x)$ has no stationary points.

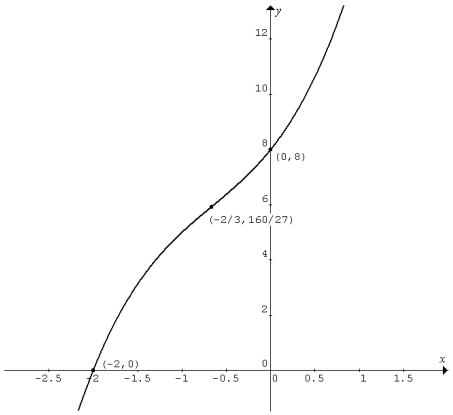
Q6aii $f''(x)=6x+4=0,\ x=-\frac{2}{3}$.
The graph of $y=f(x)$ is concave down for $x<-\frac{2}{3}$, and it is concave up for $x>-\frac{2}{3}$.

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Q6aiii x-intercept: Let $y=0$, $(x+2)(x^2+4)=0$, $\therefore\ x=-2$

y-intercept: Let $x=0$, $y=(0+2)(0^2+4)=8$



Q6bi $9=5\angle POQ$, $\therefore\ \angle POQ=\dfrac{9}{5}=1.8$

Q6bii $OP=OQ=radius$

OT is a common side for ΔOPT and ΔOQT

$\angle OPT=\angle OQT=90°$, $\therefore\ PT=QT$ (Pythagoras)

$\therefore\ \Delta OPT$ and ΔOQT are congruent because they have equal corresponding sides.

Q6biii $\angle POT=\dfrac{1.8}{2}=0.9$, length of $PT=5\tan(0.9)\approx 6.30$ cm

Q6biv Area of the shaded region

$=5\times 6.3-\dfrac{1.8}{2\pi}\times\pi\times 5^2\approx 9.00\ \text{cm}^2$

Q7ai $\ddot{x}=4\cos 2t$, $\dot{x}=1$ and $x=0$ when $t=0$

$\therefore\ \dot{x}=\displaystyle\int_0^t 4\cos 2t\,dt+1=\left[2\sin 2t\right]_0^t+1=2\sin 2t+1$

Q7aii Let $\dot{x}=2\sin 2t+1=0$,

$\sin 2t=-0.5$, $2t=\dfrac{7\pi}{6}$, $\therefore\ t=\dfrac{7\pi}{12}$

Q7aiii $\dot{x}=2\sin 2t+1$,

$x=\displaystyle\int_0^t(2\sin 2t+1)dt=\left[-\cos 2t+t\right]_0^t=1-\cos 2t+t$

Q7bi $y=x^2$: At $A(-1,1)$, $m=\dfrac{dy}{dx}=2x=-2$

Equation of the tangent: $y-1=-2(x+1)$, $2x+y=-1$

Q7bii Midpoint of $AB=M\left(\dfrac{-1+2}{2},\dfrac{1+4}{2}\right)=M\left(\dfrac{1}{2},\dfrac{5}{2}\right)$

Gradient of $AB=\dfrac{4-1}{2-(-1)}=1$

At C, $m=\dfrac{dy}{dx}=2x=1$, $\therefore\ x=\dfrac{1}{2}$

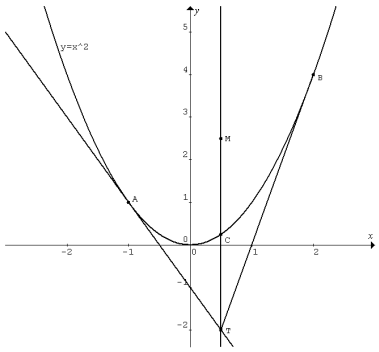
M and C have the same x-coordinate, $\therefore$ MC is vertical.

Q7biii At $B(2,4)$, $m=\dfrac{dy}{dx}=2x=4$

Tangent at A: $2x+y=-1$; line MC: $x=\dfrac{1}{2}$

$\therefore$ intersection $T\left(\dfrac{1}{2},-2\right)$

Gradient of $BT=\dfrac{4-(-2)}{2-\frac{1}{2}}=\dfrac{6}{\frac{3}{2}}=4$, same as the gradient of the tangent at B, $\therefore$ line BT is a tangent to the parabola.



Q8a Let 1935 be $t=0$, and P is in millions.

$t=0$, $P=0.000102$; $t=75$, $P=200$; $\dfrac{dP}{dt}=kP$

$\dfrac{dt}{dP}=\dfrac{1}{k}\times\dfrac{1}{P}$, $t=\dfrac{1}{k}\displaystyle\int\frac{1}{P}dP$, $kt=\ln P+c$

$t=0$, $P=0.000102$, $\therefore\ 0=\ln 0.000102+c$, $c=-\ln 0.000102$

$\therefore\ kt=\ln\dfrac{P}{0.000102}$

$t=75$, $P=200$, $\therefore\ 75k=\ln\dfrac{200}{0.000102}$, $k=0.193185$

$\therefore\ 0.193185t=\ln\dfrac{P}{0.000102}$

$\therefore\ P=0.000102e^{0.193185t}$

In 2035, $t=100$, $P\approx 25033$ millions

Q8b $\Pr(2heads)=[\Pr(head)]^2=0.36$

$\therefore\Pr(head)=0.6$, $\Pr(tail)=1-0.6=0.4$

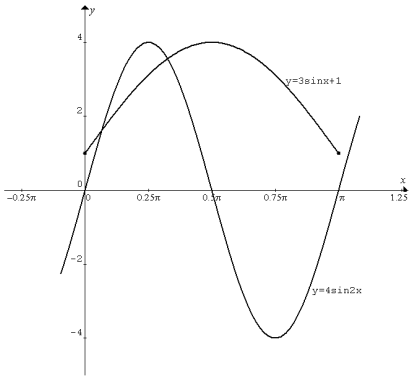
$\therefore\ \Pr(2tails)=[\Pr(tail)]^2=0.16$

Q8ci $y=A\sin bx$, amplitude $A=4$

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Q8cii Period $T=\dfrac{2\pi}{b}=\pi$, $b=2$

Q8ciii



Q8d $f(x)=x^3-3x^2+kx+8$, $f'(x)=3x^2-6x+k$

$f(x)$ is an increasing function, $f'(x)>0$, when the discriminant of $3x^2-6x+k$ is less than 0, i.e. $\Delta=36-12k<0$, $k>3$

Q9ai The balance after n deposits is given by

$B_n=\dfrac{Q(R^n-1)}{R-1}$ where $Q=\$500$ and $R=1.005$

$\therefore\ B_{240}=\dfrac{500(1.005^{240}-1)}{0.005}=231020.4476$

$\therefore$ at the end of the 240th month,

$\$P=231020.4476\times1.005=\$232175.55$

Q9aii(1) $A_n=PR^n-\dfrac{Q(R^n-1)}{R-1}$ where $Q=\$2000$ and $R=1.005$

$\therefore\ A_n=P\times1.005^n-\dfrac{2000(1.005^n-1)}{0.005}$

$=P\times1.005^n-400000(1.005^n-1)$

$=(P-400000)\times1.005^n+400000$

Q9aii(2) Let $A_n=(P-400000)\times1.005^n+400000=0$

$\therefore\ 1.005^n=\dfrac{-400000}{P-400000}=\dfrac{-400000}{232175.55-400000}=2.383443$

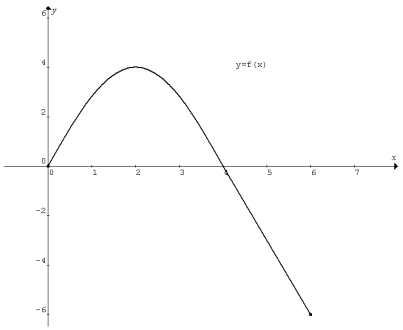
$n=\dfrac{\ln 2.383443}{\ln 1.005}\approx174.14$, $\therefore$ there will be money left in the account 174 months after retirement.

Q9bi $f'(x)>0$, i.e. $f(x)$ is increasing for $0\le x<2$

Q9bii Given $f(0)=0$, max $f(x)=f(2)=\int_0^2 f'(x)dx=A_1=4$

Q9biii $f(6)=A_1-A_2-3\times2=-6$

Q9biv



Q10ai $\angle DAC=\angle CAB$ (common angle)

$\therefore\ \angle DCA=\angle CBA$ (both triangles are isosceles)

$\therefore\ \angle ADC=\angle ACB=\theta$

$\therefore\ \Delta ABC$ and ΔACD are similar.

Q10aii ΔABC and ΔACD are similar,

$\therefore\ \dfrac{x}{a}=\dfrac{a+y}{x}$, $x^2=a(a+y)$, $\therefore\ x^2=a^2+ay$

Q10aiii Apply the cosine rule to ΔABC:

$(a+y)^2=x^2+x^2-2x^2\cos\theta$

$\therefore\ (a+y)^2=2x^2(1-\cos\theta)$, $(a+y)^2=2a(a+y)(1-\cos\theta)$

$a+y=2a(1-\cos\theta)$, $\therefore\ y=a(1-2\cos\theta)$

Q10aiv Maximum of $1-2\cos\theta$ is 3 when $\theta=-\pi$, $\therefore\ y\le3a$

Q10bi $x^2+y^2=r^2$, $\therefore\ y^2=r^2-x^2$

$$V=\int_{r\sin\theta}^{r}\pi y^2dx=\int_{r\sin\theta}^{r}\pi(r^2-x^2)dx=\pi\left[r^2x-\frac{x^3}{3}\right]_{r\sin\theta}^{r}$$

$$=\pi\left(\frac{2r^3}{3}-r^3\sin\theta+\frac{r^3\sin^3\theta}{3}\right)=\frac{\pi r^3}{3}(2-3\sin\theta+\sin^3\theta)$$

Q10ii(1) Original depth $=r$ m, final depth $=\dfrac{r}{2}$ m at angle θ

$\therefore\ r\sin\theta=\dfrac{r}{2}$, $\theta=\dfrac{\pi}{6}$

Q10ii(2) Original volume $=\dfrac{1}{2}\times\dfrac{4\pi r^3}{3}=\dfrac{2\pi r^3}{3}$

Final volume $=\dfrac{\pi r^3}{3}\left(2-3\sin\dfrac{\pi}{6}+\sin^3\dfrac{\pi}{6}\right)=\dfrac{\pi r^3}{3}\left(\dfrac{5}{8}\right)$

Fraction $=\dfrac{\frac{5}{8}\times\frac{\pi r^3}{3}}{\frac{2\pi r^3}{3}}=\dfrac{5}{16}$

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