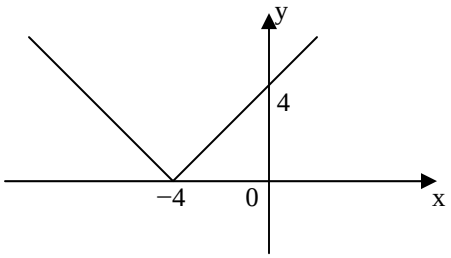
NSW BOS Mathematics Solutions 2006
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Q1a $e^{-0.5} = 0.607$

Q1b $2x^2 + 5x - 3 = (2x-1)(x+3)$

Q1c



Q1d $\dfrac{\sin\theta}{5} = \dfrac{\sin 33^0}{9}$, $\theta = 18^0$

Q1e $3 - 5x \le 2$, $-5x \le -1$, $x \ge \dfrac{-1}{-5}$, $x \ge \dfrac{1}{5}$

Q1f $a = \dfrac{13}{5}$, $r = \dfrac{1}{5}$, $S_\infty = \dfrac{\frac{13}{5}}{1-\frac{1}{5}} = \dfrac{13}{4}$.

Q2ai Let $y = x\tan x$, $\dfrac{dy}{dx} = \tan x + x\sec^2 x$

Q2aii Let $y = \dfrac{\sin x}{x+1} = \dfrac{(x+1)\cos x - \sin x}{(x+1)^2}$

Q2bi $\int (1 + e^{7x})dx = x + \dfrac{1}{7}e^{7x} + c$

Q2bii Let $u = 1 + x^2$, $\dfrac{du}{dx} = 2x$. When $x = 0$, $u = 1$. When $x = 3$, $u = 10$.

$$\int_0^3 \frac{8x}{1+x^2}dx = \int_0^3 \frac{4}{u}\frac{du}{dx}dx = \int_1^{10} \frac{4}{u}du = \left[4\log_e u\right]_1^{10} = 4\log_e 10.$$

Q2c $y = \cos 2x$, $\dfrac{dy}{dx} = -2\sin 2x$.

When $x = \dfrac{\pi}{6}$, $y = \dfrac{1}{2}$, $\dfrac{dy}{dx} = -\sqrt{3}$.

Equation of tangent: $y - \dfrac{1}{2} = -\sqrt{3}\left(x - \dfrac{\pi}{6}\right)$,

$y = -\sqrt{3}x + \dfrac{\pi\sqrt{3}}{6} + \dfrac{1}{2}$.

Q3ai $m = \dfrac{-4-4}{5-1} = -2$, $\therefore y = -2x + c$.

Use $(1,4)$ to find c: $4 = -2(1) + c$, $c = 6$, $\therefore y = -2x + 6$.

Q3aii $D(0,6)$

Q3aiii Let $P(a,b)$ be a point on line AB such that $CP \perp AB$.

$\therefore 2a + b - 6 = 0$, and $\dfrac{b-(-1)}{a-(-3)} \times (-2) = -1$. Solve the two equations to find $a = \dfrac{11}{5}$ and $b = \dfrac{8}{5}$.

Perpendicular distance

$$CP = \sqrt{\left(\frac{11}{5} - (-3)\right)^2 + \left(\frac{8}{5} - (-1)\right)^2} = \frac{13\sqrt{5}}{5}.$$

Q3aiv $AD = \sqrt{(1-0)^2 + (4-6)^2} = \sqrt{5}$

Area of $\Delta ADC = \dfrac{1}{2} \times \sqrt{5} \times \dfrac{13\sqrt{5}}{5} = \dfrac{13}{2}$ square units.

Q3b $\displaystyle\sum_{r=2}^{4} \frac{1}{r} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$

Q3ci $a = 560$, $d = -17$, $t_{14} = 560 + (14-1)(-17) = 339$

Q3cii $S_{14} = \dfrac{14}{2}(560 + 339) = 6293$

Q3ciii $t_n < 60$, $560 + (n-1)(-17) < 60$, $(n-1)(-17) < -500$,

$\therefore n - 1 > \dfrac{-500}{-17} = 29.4$, $n > 30.4$, i.e. $n = 31$, 31st day.

Q4ai $\angle DAB = \pi - 2\left(\dfrac{\pi}{6}\right) = \dfrac{2\pi}{3}$

Q4aii The cosine rule:

$$BD = \sqrt{3^2 + 3^2 - 2(3)(3)\cos\frac{2\pi}{3}} = 3\sqrt{3} \text{ m}$$

Q4aiii Area $ABCD$ = area of triangle ABD + area of sector BCD

$$= \frac{1}{2}(3)(3)\sin\frac{2\pi}{3} + \frac{1}{2}(3)^2\left(\frac{5\pi}{6}\right) = \frac{9}{2}\left(\frac{\sqrt{3}}{2} + \frac{5\pi}{6}\right) \text{ m}^2$$

Q4b $y = x^2 + 1$, $x^2 = y - 1$.

$$V = \int_1^5 \pi x^2 dy = \int_1^5 \pi(y-1)dy = \left[\frac{\pi(y-1)^2}{2}\right]_1^5 = 8\pi \text{ cubic units}$$

Q4ci $\Pr(3w) = \dfrac{\binom{32}{3}}{\binom{64}{3}} = \dfrac{5}{42}$

Q4cii $\Pr(3w \cup 3b) = \Pr(3w) + \Pr(3b) = \dfrac{5}{42} + \dfrac{5}{42} = \dfrac{5}{21}$

Q4ciii $\Pr(notsamecolour) = 1 - \Pr(3w \cup 3b) = 1 - \dfrac{5}{21} = \dfrac{16}{21}$

Q5ai $f(x) = 2x^2(3-x) = 6x^2 - 2x^3$, $f'(x) = 12x - 6x^2$.
At T.P., $f'(x) = 0$, $12x - 6x^2 = 0$, $6x(2-x) = 0$.
$\therefore x = 0$ and $y = 0$, or $x = 2$ and $y = 8$.
The turning points are $(0,0)$, $(2,8)$.
Nature:

x	-1	0	1	2	3
$f'(x)$	$-$	0	$+$	0	$-$

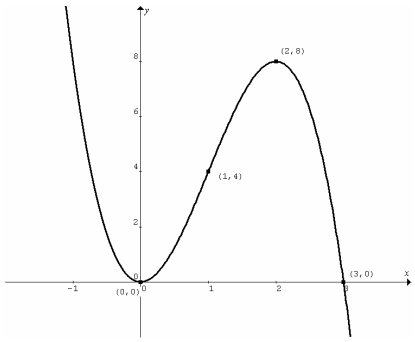
$(0,0)$ local minimum; $(2,8)$ local maximum.

Q5aii $f''(x) = 12 - 12x$. At inflexion point, $f''(x) = 0$,
$12 - 12x = 0$, $x = 1$ and $y = f(1) = 4$.

x	0.5	1	1.5
$f'(x)$	+ 4.5	+ 6	+ 4.5

The inflexion point is $(1,4)$.

Q5aiii x-intercept, let $f(x) = 0$, $2x^2(3-x) = 0$, $x = 0$, $x = 3$.



Q5aiv Minimum value of $f(x)$ for $-1 \le x \le 4$ occurs at $x = 4$,
minimum value is $f(4) = 2(4)^2(3-4) = -32$.

Q5bi $\dfrac{d}{dx}\log_e(\cos x) = \dfrac{1}{\cos x} \times (-\sin x) = -\tan x$.

Q5bii $\text{Area} = \displaystyle\int_0^{\frac{\pi}{4}} (\tan x - x)dx = \int_0^{\frac{\pi}{4}} \tan x dx - \int_0^{\frac{\pi}{4}} x dx$

$= \left[-\log_e(\cos x)\right]_0^{\frac{\pi}{4}} - \left[\dfrac{x^2}{2}\right]_0^{\frac{\pi}{4}} = -\log_e\left(\cos\dfrac{\pi}{4}\right) - \dfrac{1}{2}\left(\dfrac{\pi}{4}\right)^2$

$= \log_e\sqrt{2} - \dfrac{\pi^2}{32}$.

Q6ai Given $\angle BAC = \angle CAD$ and $BC \,//\, AD$.
$\angle CAD = \angle BCA$ (Alternate angles), $\therefore \angle BAC = \angle BCA$.

Q6aii Since $\angle BAC = \angle BCA$ and given $\angle ABP = \angle PBC$,
$\therefore \angle BPA = \angle BPC$. $\therefore \Delta BPA$ and ΔBPC are similar. Since the corresponding sides are the same side BP, $\therefore \Delta BPA \equiv \Delta BPC$.

Q6aiii Consider ΔBPA and ΔDPC.
Since $\Delta BPA \equiv \Delta BPC$, $\therefore AP = PC$.
Use similar proof as Q6aii, $\Delta BPA \equiv \Delta DPA$, $\therefore BP = PD$.
$\angle BPA = \angle DPC$ (Vertically opposite angles). Since ΔBPA and ΔDPC have two pairs of equal corresponding sides and equal included angles, $\therefore \Delta BPA \equiv \Delta DPC$.
Hence $\Delta BPA \equiv \Delta BPC \equiv \Delta DPA \equiv \Delta DPC$, and the corresponding sides $AB = BC = CD = DA$. $\therefore ABCD$ is a rhombus.

Q6bi $P = 150 + 300e^{-0.05t}$. When $t = 0$, $P = 150 + 300 = 450$.

Q6bii $\dfrac{dP}{dt} = -15e^{-0.05t}$. When $t = 10$, $\dfrac{dP}{dt} = -15e^{-0.5}$.

Q6biii As $t \to \infty$, $P \to 150$.

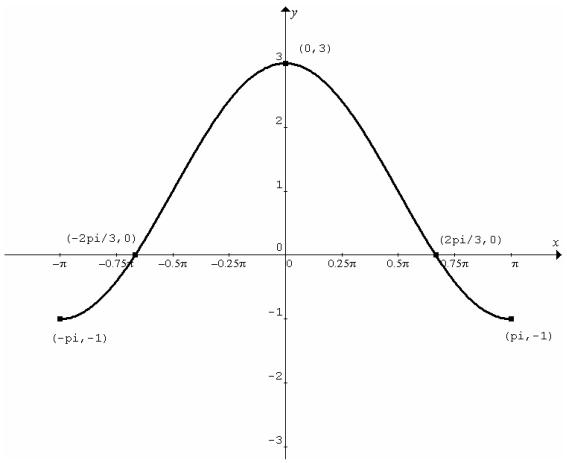
Q6biv $P < 200$, $150 + 300e^{-0.05t} < 200$, $300e^{-0.05t} < 50$,
$e^{-0.05t} < \dfrac{1}{6}$, $-0.05t < \log_e\dfrac{1}{6}$, $t > \dfrac{\log_e\frac{1}{6}}{-0.05}$, $t > 35.84$, i.e. 35.84 years after observations began.

Q7ai $x^2 - 3x + 1 = (x-\alpha)(x-\beta) = x^2 - (\alpha+\beta) + \alpha\beta$.
$\therefore \alpha\beta = 1$ and $\alpha + \beta = 3$

Q7aii $\alpha\beta = 1$, $\therefore \beta = \dfrac{1}{\alpha}$. $\therefore \alpha + \dfrac{1}{\alpha} = \alpha + \beta = 3$.

Q7bi $f(x) = 1 + 2\cos x$.
Let $f(x) = 0$, $1 + 2\cos x = 0$, $\cos x = -\dfrac{1}{2}$, $x = \dfrac{2\pi}{3}$ is one of the x-intercepts.

Q7bii



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Q7biii $\int_{-\frac{\pi}{2}}^{\frac{2\pi}{3}} (1+2\cos x)dx = \left[x+2\sin x\right]_{-\frac{\pi}{2}}^{\frac{2\pi}{3}}$

$= \frac{2\pi}{3} + 2\sin\frac{2\pi}{3} - \left(-\frac{\pi}{2}\right) - 2\sin\left(-\frac{\pi}{2}\right) = \frac{2\pi}{3} + \sqrt{3} + \frac{\pi}{2} + 2$

$= \frac{7\pi}{6} + \sqrt{3} + 2$

Q7ci $2x^2 + (k-2)x + 8 = 0$,

$\Delta = (k-2)^2 - 4(2)(8) = (k-2)^2 - 64$.

Q7cii $2x^2 + kx + 9 = 2x + 1$, $2x^2 + (k-2)x + 8 = 0$.

No intersections: $\Delta < 0$, $(k-2)^2 - 64 < 0$,

i.e. $(k-10)(k+6) < 0$, $\therefore -6 < k < 10$.

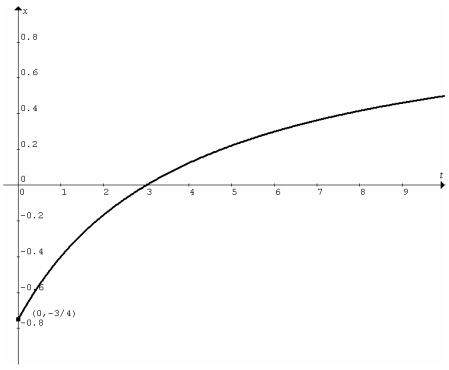
Q8ai $x = 1 - \frac{7}{t+4}$. When $t = 0$, $x = -\frac{3}{4}$.

Q8aii $v = \frac{dx}{dt} = \frac{7}{(t+4)^2}$. When $x = 0$, $1 - \frac{7}{t+4} = 0$, $t = 3$,

$v = \frac{7}{7^2} = \frac{1}{7}$.

Q8aiii $a = \frac{dv}{dt} = -\frac{14}{(t+4)^3} < 0$ for $t \geq 0$.

Q8aiv



Q8bi

$A_n = 200000r^n - M\left(1 + r + r^2 + ... + r^{n-1}\right) = 200000r^n - \frac{M\left(r^n - 1\right)}{r-1}$

$r = 1.006$, $A_{300} = 0$, $200000(1.006)^{300} - \frac{M\left(1.006^{300} - 1\right)}{1.006 - 1} = 0$,

$M = \$1439.18$

Q8bii $200000(1.006)^n - \frac{2800\left(1.006^n - 1\right)}{1.006 - 1} = 0$.

Expand and simplify:

$0.006(200000)(1.006)^n - 2800(1.006)^n + 2800 = 0$,

$1600(1.006)^n = 2800$, $n = 93.55$, i.e. 94 months.

Q9a Express $12y = x^2 - 6x - 3$ in the form $4a(y-h) = (x-k)^2$:

$12y = x^2 - 6x + 9 - 9 - 3$, $12y + 12 = x^2 - 6x + 9$,

$12(y+1) = (x-3)^2$, $\therefore a = 3$.

The vertex is $(h,k) = (3,-1)$, the focus is $(h, k+a) = (3,2)$.

Q9bi $\frac{dV}{dt} = 120 + 26t - t^2$. When $t = 0$, $\frac{dV}{dt} = 120$.

When $\frac{dV}{dt} = 2 \times 120$, $240 = 120 + 26t - t^2$, $t^2 - 26t + 120 = 0$,

$(t-6)(t-20) = 0$. Hence $t = 6$ or $t = 20$ min.

Q9bii $\Delta V(t) = \int_0^t (120 + 26t - t^2)dt = \left[120t + 13t^2 - \frac{t^3}{3}\right]_0^t$

$= 120t + 13t^2 - \frac{t^3}{3}$.

Q9biii When the tank is just full, $\Delta V = 7000 - 1500 = 5500$ L.

When $t = 30$, $\Delta V = 120(30) + 13(30)^2 - \frac{30^3}{3} = 6300$ L.

Loss = 6300 − 5500 = 800 L.

Q9ci Cone height $x = a + \sqrt{a^2 - r^2}$, $(x-a)^2 = a^2 - r^2$,

$\therefore r^2 = a^2 - (x-a)^2$, where $x > 0$.

Cone volume $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left[a^2 - (x-a)^2\right]x$

$= \frac{\pi}{3}\left(2ax^2 - x^3\right)$.

Q9cii At turning points, $\frac{dV}{dx} = \frac{\pi}{3}\left(4ax - 3x^2\right) = 0$,

$\therefore x(4a - 3x) = 0$. Since $x > 0$, $\therefore 4a - 3x = 0$, $x = \frac{4a}{3}$.

x	a	$\frac{4a}{3}$	$2a$
$\frac{dV}{dx}$	+	0	–

$\therefore x = \frac{4a}{3}$ gives the maximum volume.

Q10a

$\int_{0.5}^{1.5} (\log_e x)^3 dx \approx \frac{0.5}{3}\left[(\log_e 0.5)^3 + 4(\log_e 1.0)^3 + (\log_e 1.5)^3\right]$

$= -0.044$

Q10bi $KL = 6 + x$, $KQ = 6 - x$, where $0 \leq x \leq 6$,

$\therefore QL^2 = KL^2 - KQ^2 = (6+x)^2 - (6-x)^2 = 24x$.

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Q10bii $\angle LMN + \angle MLN = \angle KLQ + \angle MLN = 90^0$,

$\therefore \angle LMN = \angle KLQ$. Given $\angle MNL = \angle LQK = 90^0$,

$\therefore \angle MLN = \angle LKQ$. Hence $\Delta QKL, \Delta NLM$ are similar.

$\therefore \dfrac{LM}{KL} = \dfrac{NM}{QL}$, $\dfrac{y}{6+x} = \dfrac{12}{\sqrt{24x}}$,

$\therefore y = \dfrac{12(6+x)}{\sqrt{24x}} = \dfrac{\sqrt{6}(6+x)}{\sqrt{x}}$.

Q10biii

Area of $\Delta KLM = A = \dfrac{1}{2} \times \dfrac{\sqrt{6}(6+x)}{\sqrt{x}} \times (6+x) = \dfrac{\sqrt{6}(6+x)^2}{2\sqrt{x}}$.

Q10biv $12 \le y \le 13$, $12 \le \dfrac{\sqrt{6}(6+x)}{\sqrt{x}} \le 13$, where $0 \le x \le 6$.

Square the terms: $144 \le \dfrac{6(6+x)^2}{x} \le 169$.

Expand and simplify: $144x \le 216 + 72x + 6x^2 \le 169x$.

Split into two inequalities:

$144x \le 216 + 72x + 6x^2$ and $216 + 72x + 6x^2 \le 169x$

$0 \le 216 - 72x + 6x^2$ and $216 - 97x + 6x^2 \le 0$

$\therefore 0 \le 6(6-x)^2$ and $(8-3x)(27-2x) \le 0$, i.e. $\dfrac{8}{3} \le x \le \dfrac{27}{2}$.

Since $0 \le x \le 6$, $\therefore \dfrac{8}{3} \le x \le 6$.

Q10bv $A = \dfrac{\sqrt{6}(6+x)^2}{2\sqrt{x}}$, where $\dfrac{8}{3} \le x \le 6$

Turning points: $\dfrac{dA}{dx} = \dfrac{4\sqrt{6}\sqrt{x}(6+x) - \dfrac{\sqrt{6}(6+x)^2}{\sqrt{x}}}{4x} = 0$,

$4\sqrt{6}\sqrt{x}(6+x) - \dfrac{\sqrt{6}(6+x)^2}{\sqrt{x}} = 0$,

$4\sqrt{6}x(6+x) - \sqrt{6}(6+x)^2 = 0$, $\sqrt{6}(6+x)(4x - (6+x)) = 0$,

$3\sqrt{6}(6+x)(x-2) = 0$, $x = -6$ or 2.

Both values are outside $\dfrac{8}{3} \le x \le 6$.

$\dfrac{dA}{dx}$ is positive for $x > 2$, $\therefore A(x)$ is an increasing function of x for $\dfrac{8}{3} \le x \le 6$. Hence minimum area of ΔKLM occurs when $x = \dfrac{8}{3}$. Minimum $A = \dfrac{\sqrt{6}\left(6 + \dfrac{8}{3}\right)^2}{2\sqrt{\dfrac{8}{3}}} = \dfrac{169}{3}$ square units.

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