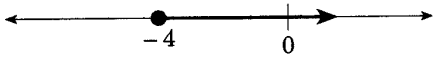
2000 HIGHER SCHOOL CERTIFICATE
SOLUTIONS
2/3 UNIT (COMMON) MATHEMATICS

QUESTION 1

(a) $\log_e 8 = 2{\cdot}079\,441\,542\ldots$
$\doteqdot 2{\cdot}08$ (2 d.p.).

(b) $x+7 \geq 3$
$x \geq -4$.



(c) $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$.

(d) P(blue) $= \frac{3}{5}$.

(e) By substitution:

$x-y = 2$ ①
$3x+2y = 1$. ②

From ①, $x = y+2$.

Substitute for x in ②:

$3(y+2)+2y = 1$
$5y+6 = 1$
$5y = -5$
$y = -1$.

Substitute for y in ①:

$x-(-1) = 2$
$x = 1$.

Solution: $x = 1,\ y = -1$.

OR By elimination:

$x-y = 2$ ①
$3x+2y = 1$. ②

Multiply ① by 2:

$2x-2y = 4$ ③
$3x+2y = 1$. ②

Add ③ + ②: $5x = 5$
$\therefore\ x = 1$.

Substitute into ①:

$1-y = 2$
$\therefore\ y = -1$.

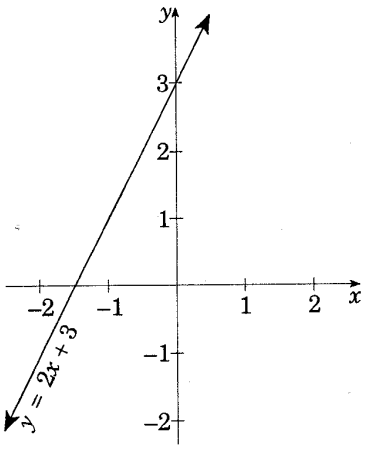
Solution: $x = 1,\ y = -1$.

(f) $|x-5| = 3$

$x-5 = 3$ or $-(x-5) = 3$
$\therefore\ x = 8$. $x-5 = -3$
$\therefore\ x = 2$.

Solution: $x = 8$ or $x = 2$.

(g)



QUESTION 2

(a) Gradient of $PQ = \frac{0-2}{4-0} = -\frac{1}{2}$.

(b) M, midpoint of PQ is $M\left(\frac{0+4}{2}, \frac{2+0}{2}\right)$.

$\therefore\ M(2, 1)$.

(c) $MN \perp PQ$,

$\therefore$ gradient of $MN = 2$ $(m_1m_2 = -1)$.

Equation of MN is

$y-1 = 2(x-2)$
$= 2x-4$
$2x-y-3 = 0$.

(d) Substitute $x = 0$ in equation of MN.

$2(0)-y-3 = 0$
$y = -3$.

$\therefore\ N$ has coordinates $(0, -3)$.

(e) $NQ = \sqrt{(4-0)^2+(0+3)^2} = 5$.

$\therefore$ distance $NQ = 5$ units.

(f) $r = 5$. Centre is $(0, -3)$.

$\therefore$ Equation of circle is $(x-0)^2 + (y+3)^2 = 5^2$

or $x^2 + (y+3)^2 = 25$.

(g) $PN = 2 - (-3)$

$= 5$

= radius of circle.

$\therefore$ Circle passes through P.

OR Substitute coordinates of P in the equation of the circle.

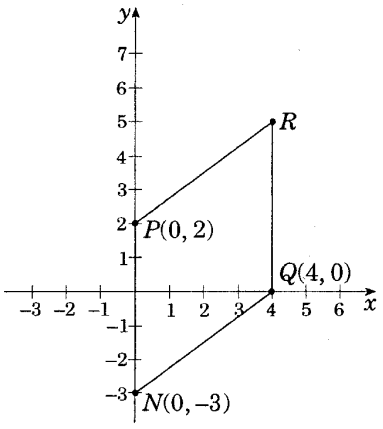
LHS $= 0^2 + (2+3)^2$

$= 25$

= RHS.

$\therefore$ Circle passes through P.

(h)



By observation: R is $(4, 5)$, since Q is 4 units to the right of PN and P is 5 units up the y axis from N.

QUESTION 3

(a) (i) $\dfrac{d}{dx}\left(3xe^x\right) = 3e^x + 3xe^x$

$= 3e^x(1+x)$.

(ii) $\dfrac{d}{dx}\left[\sin\left(x^2+1\right)\right] = 2x\cos\left(x^2+1\right)$.

(b) $\Delta ABC \parallel\!\parallel \Delta DEC$.

$\therefore \quad \dfrac{AB}{DE} = \dfrac{AC}{DC}$

$\dfrac{x}{1{\cdot}5} = \dfrac{7}{2}$

$x = 5{\cdot}25$.

$\therefore$ $AB = 5{\cdot}25$ cm.

(c) (i) $\displaystyle\int \sec^2 5x\, dx = \frac{1}{5}\tan 5x + C$.

(ii) $\displaystyle\int_{-2}^{1} \frac{2}{x+3}\, dx = 2\Big[\ln(x+3)\Big]_{-2}^{1}$

$= 2(\ln 4 - \ln 1)$

$= 2\ln 4$.

(d) $y = 2\log_e x$, $\dfrac{dy}{dx} = \dfrac{2}{x}$.

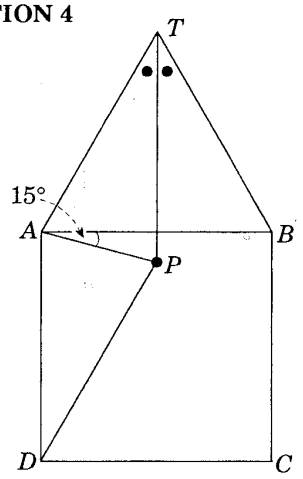
At $x = 1$, $\dfrac{dy}{dx} = 2$.

$\therefore$ Equation of the tangent is $y - 0 = 2(x-1)$

$y = 2x - 2$.

QUESTION 4

(a)



(i) $\angle TAB = 60°$ (ΔABT is equilateral.)

$\angle PAT = \angle TAB + \angle PAB$

$= 60° + 15°$

$= 75°$.

(ii) In ΔTAP and ΔDAP,

$AT = AD$ (ΔTAB equilateral; $ABCD$ square)

AP is common.

$\angle DAP = \angle DAB - \angle PAB$

$= 90° - 15°$

$= 75°$

$= \angle PAT$.

$\therefore$ $\Delta TAP \equiv \Delta DAP$ **(SAS)**.

(iii) $\angle ATB = 60°$ (ΔABT is equilateral.)

$\angle ATP = 30°$ (TP bisects $\angle APB$.)

$\angle ADP = \angle ATP$ (corresponding $\angle$s in congruent Δs)

$= 30°$.

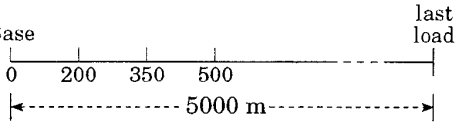
$\angle APD = 180° - 75° - 30°$ ($\angle$ sum of a Δ)

$= 75°$

$= \angle DAP$.

$\therefore$ ΔDAP is isosceles.

(b) (i)



This is an arithmetic series, with $a = 200$, $d = 150$.

$T_n = a+(n-1)d$

$\therefore\ T_{15} = 200+14\times 150$

$= 2300.$

$\therefore$ The fifteenth load is deposited 2300 m from the base.

(ii) $T_n = 5000,$

$5000 = 200+(n-1)\times 150$

$150(n-1) = 4800$

$n-1 = 32$

$n = 33.$

$\therefore$ There are 33 loads.

(iii) Using $S_n = \frac{n}{2}(a+\ell)$,

total distance travelled

$= 2\times\left[\frac{33}{2}(200+5000)\right]$

$= 171600$ m

$= 171{\cdot}600$ km.

QUESTION 5

(a) $\tan x = 2.$

1st quadrant: $x = 1{\cdot}107\ 14...$

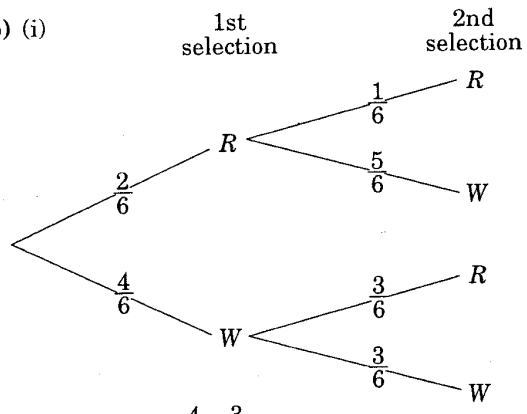
$\fallingdotseq 1{\cdot}11$ (2 d.p.).

3rd quadrant: $x = 1{\cdot}107\ 14+\pi$

$= 4{\cdot}248\ 74...$

$\fallingdotseq 4{\cdot}25$ (2 d.p.).

(b) (i)



(ii) $P(WW) = \frac{4}{6}\times\frac{3}{6}$

$= \frac{1}{3}.$

(iii) $P(RW)+P(WW) = \frac{2}{6}\times\frac{5}{6}+\frac{4}{6}\times\frac{3}{6} = \frac{11}{18}.$

(c) (i) At $t=0,\ N = 200.$

At $t=3,\ N = 400.$

$N = 200e^{kt}$

$t=3:\quad 400 = 200e^{3k}$

$e^{3k} = 2$

$3k = \ln 2.$

$\therefore\ k = \frac{1}{3}\ln 2$

$= 0{\cdot}210\ 4906$

$\fallingdotseq 0{\cdot}2310$ (4 d.p.).

(ii) At $t=12,\ N = 200e^{k\times 12}$

$= 3198{\cdot}116\ 643\ ...$

$\fallingdotseq 3200$ (2 sig. figs).

(iii) $\frac{dN}{dt} = kN.$

When $t=3,\ N=400,$

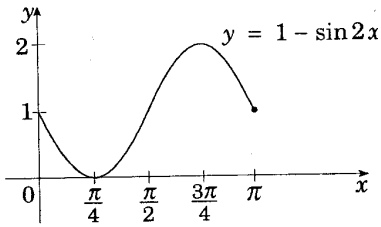
$\therefore\ \frac{dN}{dt} = k\times 400$

$\fallingdotseq 92{\cdot}4$ (3 sig. figs).

$\therefore$ The rate is 92·4 insects per week.

QUESTION 6

(a)



(b) $N = 175+18t^2-t^4,\ 0\le t\le 5.$

(i) When $t=0,\ N=175.$

$\therefore$ The initial number of students was 175.

(ii) When $t=5,\ N = 175+18(5)^2-(5)^4$

$= 0.$

$\therefore$ There were no students logged on at the end of five hours.

(iii) $\frac{dN}{dt} = 36t-4t^3$

Let $\frac{dN}{dt} = 0,\quad 36t-4t^3 = 0$

$4t(9-t^2) = 0.$

$\therefore\ t=0$ or $t=3$ or $t=-3$ (out of domain).

At $t=0,\ N=175.$

At $t=3,\ N=256.$

$\frac{d^2N}{dt^2} = 36-12t^2 < 0$ when $t=3.$

$\therefore$ Maximum value of N occurs at $t=3.$

$\therefore$ The maximum value of N is 256.

(iv) Students are logging onto the website most rapidly when $\frac{dN}{dt}$ is a maximum.

That is, when $\frac{d^2N}{dt^2} = 0$

$36t-12t^2 = 0$

$t = +\sqrt{3}$ or $-\sqrt{3}$

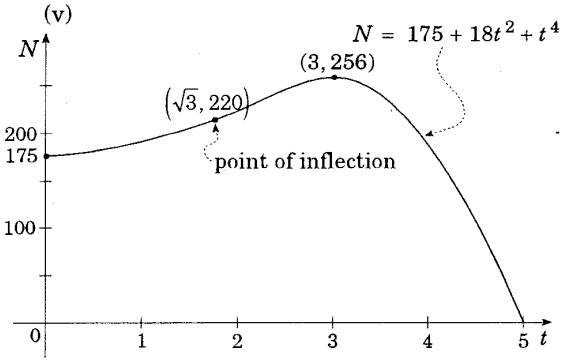
(not in domain).

Test for maximum or minimum of $\frac{dN}{dt}$:

t	$\sqrt{3}^-$	$\sqrt{3}$	$\sqrt{3}^+$
$\frac{d^2N}{dt^2}$	+	0	−

N.B. This is checking the first derivative of $\frac{dN}{dt}$.

∴ Students are logging onto the website most rapidly when $t = \sqrt{3}$ hours.



QUESTION 7

(a) $V = \pi\int_1^{e^2} y^2\,dx$

$= \pi\int_1^{e^2} \left(\frac{1}{\sqrt{x}}\right)^2 dx$

$= \pi\int_1^{e^2} \frac{1}{x}\,dx$

$= \pi\left[\ln x\right]_1^{e^2}$

$= \pi\left(\ln e^2 - \ln 1\right)$

$= \pi(2\ln e - \ln 1)$.

∴ Volume $= 2\pi$ unit^3.

(b)

x	0	0·5	1
y	sin 1	sin 1·25	sin 2

$\int_0^1 \sin\left(1+x^2\right) dx$

$\doteqdot \frac{1-0}{6}(\sin 1 + 4\sin 1{\cdot}25 + \sin 2)$

$\doteqdot 0{\cdot}924\,451\,148$

$\doteqdot 0{\cdot}92$ (2 d.p.).

(c) (i) $y = x^2 - 2$ ①

$y = x$. ②

Substitute for y in ①:

$x = x^2 - 2$

$x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$

$x = -1$ or $x = 2$.

∴ At P, $x = -1$ and at Q, $x = 2$.

(ii) Area $= \int_{-1}^{2} x - \left(x^2 - 2\right) dx$

$= \int_{-1}^{2} x - x^2 + 2\,dx$

$= \left[\frac{x^2}{2} - \frac{x^3}{3} + 2x\right]_{-1}^{2}$

$= \left(\frac{2^2}{2} - \frac{2^3}{3} + 2\times 2\right) - \left(\frac{1}{2} + \frac{1}{3} - 2\right)$

$= 4\frac{1}{2}$.

∴ Area is $4\frac{1}{2}$ square units.

QUESTION 8

(a) (i) $x = 2t - 3\log_e(t+1)$

$v = \frac{dx}{dt}$

$= 2 - \frac{3}{t+1}$.

(ii) Let $t = 0$, $v = 2 - 3 = -1$.

∴ Initial velocity is -1 m s^{-1}.

(iii) Let $v = 0$, $2 - \frac{3}{t+1} = 0$

$2t + 2 - 3 = 0$

$t = \frac{1}{2}$.

∴ Particle is at rest when $t = \frac{1}{2}$ s.

(iv) At $t = 0$, $x = 2(0) - 3\ln 1 = 0$.

At $t = \frac{1}{2}$, $x = 2(0{\cdot}5) - 3\ln 1{\cdot}5$

$= 1 - 3\ln 1{\cdot}5$

$= -0{\cdot}2164$.

At $t = 3$, $x = 2\times 3 - 3\ln 4$

$= 6 - 3\ln 4$

$= 1{\cdot}8411$.

A	O	B
$t = \frac{1}{2}$	$t = 0$	$x = 1{\cdot}8411$
$x = -0{\cdot}2164$	$x = 0$	$t = 3$

Distance OA = 0·2164 m.

Distance OB = 1·8411 m.

Distance travelled = $2OA + OB$

$= 2\times 0{\cdot}2164 + 1{\cdot}8411$

$= 2{\cdot}2739\ldots$

$= 2{\cdot}27$ m (3 sig. figs).

OR At $t = 0$, $x = 0$.

At $t = \frac{1}{2}$, $x = 1 - 3\ln 1{\cdot}5$.

At $t = 3$, $x = 6 - 3\ln 4$.

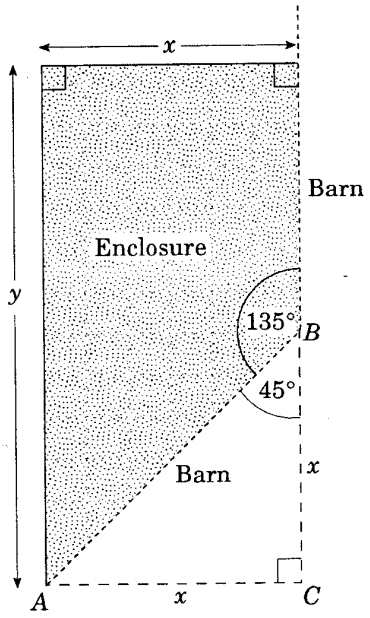
Distance travelled in the first 3 seconds

$= |1 - 3\ln 1{\cdot}5 - 0| + |6 - 3\ln 4 - (1 - 3\ln 1{\cdot}5)|$

$= 3\ln 1{\cdot}5 - 1 + 6 - 3\ln 4 - 1 + 3\ln 1{\cdot}5$

$= 2{\cdot}27$ m.

(b)



(i) Since $\angle ABC = 45°$, ABC is a right-angled isosceles triangle.

$\therefore\ AC = BC = x$.

$A = xy - \frac{1}{2}x^2$

$A = x(117 - x) - \frac{1}{2}x^2$ (since $x + y = 117$)

$A = 117x - \frac{3}{2}x^2$.

(ii) $A = 117x - \frac{3}{2}x^2$

$\frac{dA}{dx} = 117 - 3x$.

Let $\frac{dA}{dx} = 0$, $117 - 3x = 0$

$x = 39$.

Now $\frac{d^2A}{dx^2} = -3$.

$\therefore$ Area is a maximum when $x = 39$.

$x + y = 117$

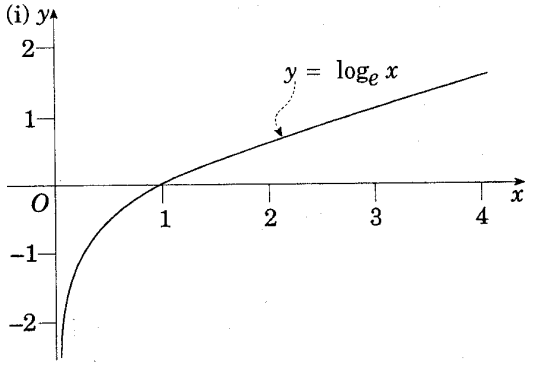
$y = 78$

$= 2 \times 39$.

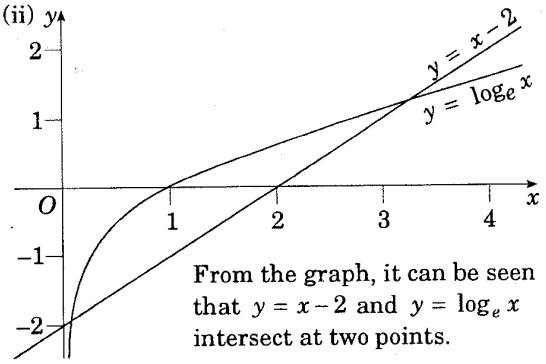
Hence area is a maximum when $y = 2x$.

QUESTION 9

(a) (i)



(ii)



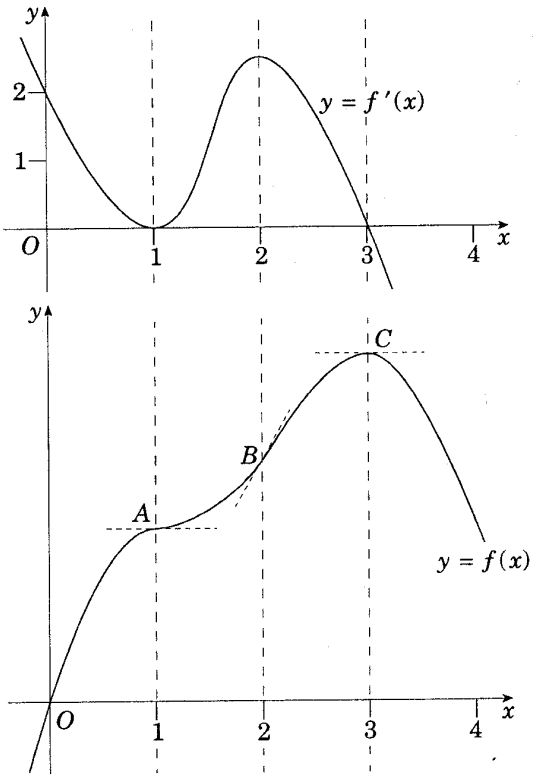
From the graph, it can be seen that $y = x - 2$ and $y = \log_e x$ intersect at two points.

$\therefore$ There are two solutions to $\log_e x = x - 2$.

That is, there are two solutions to

$\log x - x = -2$.

(b)

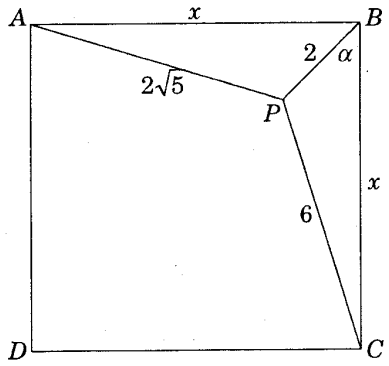


A: Horizontal point of inflection.

B: Point of inflection.

C: Maximum turning point.

(c)



NOT TO SCALE

(i) In ΔPBC, $\cos\alpha = \dfrac{PB^2 + CB^2 - PC^2}{2 \times PB \times CB}$

$= \dfrac{2^2 + x^2 - 6^2}{2 \times 2 \times x}$

$= \dfrac{x^2 - 32}{4x}$.

(ii) Since $\angle ABC$ is an angle in a square and $\angle PBC = \alpha$, then $\angle ABP = 90° - \alpha$.

Using the cosine rule in ΔABP:

$$\cos\angle ABP = \frac{x^2 + 2^2 - \left(2\sqrt{5}\right)^2}{2 \times x \times 2}$$

$$= \frac{x^2 - 16}{4x}.$$

But $\cos(90° - \alpha) = \sin\alpha$

$\therefore \quad \sin\alpha = \dfrac{x^2 - 16}{4x}$.

(iii) From (i), $\cos\alpha = \dfrac{x^2 - 32}{4x}$.

From (ii), $\sin\alpha = \dfrac{x^2 - 16}{4x}$.

But $\cos^2\alpha + \sin^2\alpha = 1$

$\therefore \left(\dfrac{x^2 - 32}{4x}\right)^2 + \left(\dfrac{x^2 - 16}{4x}\right)^2 = 1.$

$\therefore x^4 - 64x^2 + 1024 + x^4 - 32x^2 + 256 = 16x^2$

$\therefore \quad 2x^4 - 112x^2 + 1280 = 0$

$\therefore \quad x^4 - 56x^2 + 640 = 0.$

That is,

x is a solution of $x^4 - 56x^2 + 640 = 0$.

(iv) $x^4 - 56x^2 + 640 = 0$.

That is, $\left(x^2 - 16\right)\left(x^2 - 40\right) = 0$

$\therefore \quad x = \pm 4$ or $x = \pm\sqrt{40}$.

But x cm is the length of the side of a square, so $x > 0$.

$\therefore \quad x = 4$ or $\sqrt{40}\ \left(= 2\sqrt{10}\right)$.

If $x = 4$, $\cos\alpha = \dfrac{4^2 - 32}{4 \times 4} < 0$.

But α is acute, so $\cos\alpha > 0$.

If $x = 2\sqrt{10}$, $\cos\alpha = \dfrac{\left(2\sqrt{10}\right)^2 - 32}{4 \times 2\sqrt{10}} > 0.$

$\therefore$ Solution is $x = 2\sqrt{10}$.

QUESTION 10

(a) (i) $A_1 = 5000 - M.$

$A_2 = 5000 - 2M.$

$A_3 = 5000 - 3M.$

(ii) $A_4 = (5000 - 3M) \times 1{\cdot}01 - M.$

$A_5 = A_4 \times 1{\cdot}01 - M.$

$\therefore \quad A_5 = \left[(5000 - 3M) \times 1{\cdot}01 - M\right] \times 1{\cdot}01 - M$

$= (5000 - 3M) \times 1{\cdot}01^2 - M \times 1{\cdot}01 - M$

$= (5000 - 3M) \times 1{\cdot}01^2 - M(1 + 1{\cdot}01).$

(iii) $A_6 = A_5 \times 1{\cdot}01 - M$

$= (5000 - 3M) \times 1{\cdot}01^3 - M\left(1 + 1{\cdot}01 + 1{\cdot}01^2\right).$

Continuing the pattern,

$A_{36} = (5000 - 3M) \times 1{\cdot}01^{33} - M\left(1 + 1{\cdot}01 + \cdots + 1{\cdot}01^{32}\right)$

$= (5000 - 3M) \times 1{\cdot}01^{33} - M \times \dfrac{\left(1{\cdot}01^{33} - 1\right)}{(1{\cdot}01 - 1)}.$

(iv) But $A_{36} = 0$.

$\therefore (5000 - 3M) \times 1{\cdot}01^{33} - \dfrac{M \times \left(1{\cdot}01^{33} - 1\right)}{0{\cdot}01} = 0,$

$\therefore 5000 \times 1{\cdot}01^{33} - 3M \times 1{\cdot}01^{33} - M \times \dfrac{1{\cdot}01^{33} - 1}{0{\cdot}01} = 0,$

$\therefore 5000 \times 1{\cdot}01^{33} - M\left(3 \times 1{\cdot}01^{33} + \dfrac{1{\cdot}01^{33} - 1}{0{\cdot}01}\right) = 0,$

$\therefore M = \dfrac{5000 \times 1{\cdot}01^{33}}{\left(3 \times 1{\cdot}01^{33} + \dfrac{1{\cdot}01^{33} - 1}{0{\cdot}01}\right)},$

$\therefore M = 161{\cdot}343\,9692.$

$\therefore$ The monthly repayment is \$161·34 to the nearest cent.

(b) (i) The depth of the snow increases at a constant rate.

Let $\dfrac{dh}{dt} = c$, where c is a constant.

Then $h = ct + d$, where d is a constant.

But at $t = 0$, $h = 0$,
$\therefore$ $h = ct$.

It is given that $v = \dfrac{A}{h}$, where A is a constant.

Since $v = \dfrac{dx}{dt}$ and $h = ct$,

$$\frac{dx}{dt} = \frac{A}{ct}$$
$$= \frac{k}{t}, \text{ where } k = \frac{A}{c}.$$

(ii) *Without calculus:*

- 1st km of road takes 2 hours to clear (6 am to 8 am),
- 2nd km of road takes $3\frac{1}{2}$ hours.

This is a difference of $1\frac{1}{2}$ hours.

This means that it took an extra $1\frac{1}{2}$ hours to clear the snow which fell during the 2nd km clearing time; that is, it took an extra $1\frac{1}{2}$ hours to clear $3\frac{1}{2}$ hours of snow.

This is a rate of 1 hour to clear

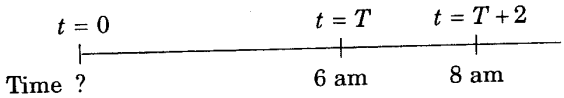
$$\frac{3\frac{1}{2}}{1\frac{1}{2}} = 2\tfrac{1}{3} \text{ hours of snow.}$$

$\therefore$ The first km taking 2 hours must have cleared $2 \times 2\frac{1}{3}$ hours, that is, $4\frac{2}{3}$ hours of snow. This time finished at 8 am.

$\therefore$ The snow started falling at $4\frac{2}{3}$ hours before 8 am. That is, it began to snow at 3:20 am.

OR x = distance in km that the snow plough has cleared.

t = time in hours that the snow has been falling.



$$\frac{dx}{dt} = \frac{k}{t}$$
$$\therefore \quad x = \int \frac{k}{t}\, dt.$$

From 6 am (time T) to 8 am (time $T + 2$), 1 km of snow was cleared.

Using $x = \int \dfrac{k}{t}\, dt$, $\quad 1 = \displaystyle\int_T^{T+2} \frac{k}{t}\, dt.$

Similarly in the next 3·5 hours, that is, from $t = T + 2$ to $t = T + 5{\cdot}5$, another 1 km of snow was cleared.

$$\therefore \quad 1 = \int_{T+2}^{T+5\cdot5} \frac{k}{t}\, dt.$$

$$\therefore \quad \int_T^{T+2} \frac{k}{t}\, dt = \int_{T+2}^{T+5\cdot5} \frac{k}{t}\, dt,$$

that is, $\quad k\Big[\ln t\Big]_T^{T+2} = k\Big[\ln t\Big]_{T+2}^{T+5\cdot5}$

that is, $\quad k\big[\ln(T+2) - \ln T\big]$
$$= k\big[\ln(T+5{\cdot}5) - \ln(T+2)\big].$$

$$\therefore \quad \ln\left(\frac{T+2}{T}\right) = \ln\left(\frac{T+5\cdot5}{T+2}\right)$$

$$\therefore \quad \frac{T+2}{T} = \frac{T+5\cdot5}{T+2}$$

$$\therefore \quad (T+2)^2 = T(T+5{\cdot}5)$$

$$\therefore \quad T = 2\tfrac{2}{3}.$$

$\therefore$ The snow started falling $2\frac{2}{3}$ hours before time T (6 am).

$\therefore$ The snow started at 3:20 am.

END OF 2/3 UNIT (COMMON) MATHEMATICS SOLUTIONS