1999 HIGHER SCHOOL CERTIFICATE SOLUTIONS

2/3 UNIT (COMMON) MATHEMATICS

QUESTION 1

(a) $A(3,-4)$, $B(7,2)$

$$\text{Midpoint} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$
$$= \left(\frac{3+7}{2}, \frac{-4+2}{2}\right)$$
$$= (5,-1).$$

(b) $e^3 = 20{\cdot}085\,536\,92 = 20{\cdot}1$ (3 sig. figs).

(c) $3-2x \geq 7$

$-2x \geq 4$

$x \leq -2.$

(d)
$$x+y = 1 \quad \text{—①}$$
$$2x-y = 5 \quad \text{—②}$$

① + ②: $3x = 6$

$x = 2.$

Substitute in ①:

$2+y = 1$

$y = -1.$

Solution: $x = 2,\ y = -1.$

(e)
$$\left(5-\sqrt{2}\right)^2 = a+b\sqrt{2}$$
$$\left(5-\sqrt{2}\right)^2 = (5)^2 - 2\times5\times\sqrt{2} + \left(\sqrt{2}\right)^2$$
$$= 25-10\sqrt{2}+2$$
$$= 27-10\sqrt{2}.$$

$\therefore\ 27-10\sqrt{2} = a+b\sqrt{2}$

$a = 27,\ b = -10.$

(f) $\text{Area} = \frac{1}{2}r^2\theta = \frac{1}{2}\times20^2\times\frac{\pi}{5}$
$= 40\pi \text{ cm}^2$
$= 126 \text{ cm}^2 \ (\text{nearest cm}^2).$

QUESTION 2

(a) (i) $\int\left(\frac{1}{x^2}+\frac{1}{x}\right)dx = \int\left(x^{-2}+\frac{1}{x}\right)dx$
$= \frac{x^{-1}}{-1} + \ln x + c$
$= \frac{-1}{x} + \ln x + c.$

(ii) $\int\cos(2x+1)\,dx = \frac{1}{2}\int 2\cos(2x+1)\,dx$
$= \frac{1}{2}\sin(2x+1)+C.$

(b) (i) $\text{Gradient } AB = \frac{5-0}{3-(-2)} = \frac{5}{5} = 1.$

(ii) $y-y_1 = m(x-x_1)$
$y-0 = 1[x-(-2)]$
$y = x+2.$

(iii) $\tan\angle BAC = \text{gradient } AB = 1.$
$\therefore\ \angle BAC = 45°.$

OR In ΔABD, $\angle D = 90°$, $AD = 5$, $BD = 5$.
$\therefore\ \Delta ABD$ is isosceles and right angled,
$\therefore\ \angle BAC = 45°.$

(iv) BD equals 5; BC, the hypotenuse of ΔBDC, equals 13; so by Pythagoras, DC equals 12.

(v) $\text{Area } \Delta ABC = \frac{1}{2}\times17\times5 \text{ sq. units}$
$= 42{\cdot}5 \text{ sq. units.}$

(vi) $\cos\angle CBD = \frac{5}{13}$
$\therefore\ \angle CBD = 67°$
$\therefore\ \angle ABC = 45°+67°$
$= 112°.$

OR $\frac{\sin\angle ABC}{17} = \frac{\sin\angle BAC}{13}$
$\sin\angle ABC = \frac{17\sin45}{13}$
$= \frac{17}{13\sqrt{2}}.$
$\therefore\ \angle ABC = 68° \text{ or } 180°-68°.$
But $\angle ABC$ is obtuse,
$\therefore\ \angle ABC = 180°-68°$
$= 112°.$

QUESTION 3

(a) (i) $\dfrac{d}{dx}(x\tan x) = \tan x \times 1 + x \times \sec^2 x$

$= \tan x + x\sec^2 x.$

(ii) $\dfrac{d}{dx}\left(\dfrac{e^x}{1+x}\right) = \dfrac{(1+x)\times e^x - e^x \times 1}{(1+x)^2}$

$= \dfrac{e^x + xe^x - e^x}{(1+x)^2}$

$= \dfrac{xe^x}{(1+x)^2}.$

(b) $y = \sqrt{x+2} = (x+2)^{\frac{1}{2}}.$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{2}(x+2)^{-\frac{1}{2}} \times 1$

$= \dfrac{1}{2\sqrt{x+2}}.$

Let m_1 = gradient of tangent at (7, 3)

and m_2 = gradient of normal at (7, 3).

$\therefore \quad m_1 = \dfrac{1}{2\sqrt{7+2}}$

$= \dfrac{1}{2\sqrt{9}}$

$= \dfrac{1}{6}.$

Now $m_2 = -\dfrac{1}{m_1}$ (for $\perp$ lines)

$= -6.$

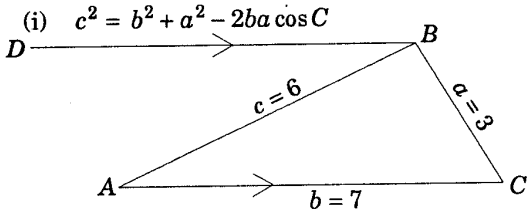
Equation of normal is $y - y_1 = m_2(x - x_1)$

$\therefore \quad y - 3 = -6(x-7)$

$= -6x + 42$

$\therefore \quad y = -6x + 45$ or $6x + y - 45 = 0.$

(c) (i) $c^2 = b^2 + a^2 - 2ba\cos C$



$6^2 = 7^2 + 3^2 - 2\times 7 \times 3\cos C$

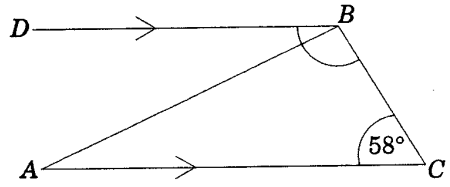
$\cos C = \dfrac{7^2 + 3^2 - 6^2}{2\times 7\times 3}$

$\therefore \quad \angle C = 58{\cdot}411\,864\,49°$

$= 58°$ (nearest degree).

That is, $\angle ACB = 58°$.

(ii)



$\angle DBC + 58° = 180°$ (cointerior $\angle$s, $AC \| DB$)

$\therefore \quad \angle DBC = 180° - 58°$

$= 122°.$

QUESTION 4

(a) $a = 8, \quad S = 12$

Now $S = \dfrac{a}{1-r}, \quad \therefore \quad 12 = \dfrac{8}{1-r}$

$12(1-r) = 8$

$12 - 12r = 8$

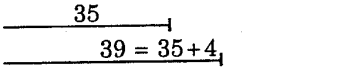
$-12r = -4$

$r = \dfrac{-4}{-12}$

$= \dfrac{1}{3}.$

That is, the common ratio is $\dfrac{1}{3}$.

(b)



This is an arithmetic sequence with $a = 35$, $d = 4$.

(i) $T_n = a + (n-1)d$

$T_{12} = 35 + 11\times 4$

$= 79.$

(ii) $T_n > 200, \qquad n = ?$

$\therefore \quad a + (n-1)d > 200$

$35 + (n-1)\times 4 > 200$

$35 + 4n - 4 > 200$

$4n > 169$

$n > 42\frac{1}{4}.$

Since n is a whole number, the 43rd row will be the first row to contain more than 200 cabbages.

(iii) $S_n = 945, \qquad S_n = \dfrac{n}{2}[2a + (n-1)d]$

$\therefore \quad 945 = \dfrac{n}{2}[2\times 35 + (n-1)\times 4]$

$= \dfrac{n}{2}(70 + 4n - 4)$

$= \dfrac{n}{2}(66 + 4n)$

$= 33n + 2n^2.$

$\therefore 2n^2 + 33n - 945 = 0$

$(2n + 63)(n - 15) = 0$

$\therefore \quad n = 15$ or $-31{\cdot}5.$

But $n > 0$, so $n = 15$.

That is, 15 rows are needed for a total of 945 cabbages.

(c) (i) $y = 4x - x^2$

At B, $y = 0, \quad \therefore \quad 4x - x^2 = 0$

$x(4-x) = 0$

$x = 0$ or $4.$

But $x = 0$ is the origin,

$\therefore$ at B, $x = 4$.

(ii) Area $= \int_0^4 \left(4x - x^2\right)\,dx$

$= \left[2x^2 - \frac{x^3}{3}\right]_0^4$

$= \left(2\times 4^2 - \frac{4^3}{3}\right) - (0)$

$= \left(32 - 21\frac{1}{3}\right)$

$= 10\frac{2}{3}$ square units.

(iii) $y \geq 0,\quad y \leq 4x - x^2.$

(N.B. $y > 0,\ y < 4x - x^2$ is also correct.)

QUESTION 5

(a) $y = x^3 - 6x^2 + 9x + 1$

(i) $\frac{dy}{dx} = 3x^2 - 12x + 9$

$= 3\left(x^2 - 4x + 3\right).$

(ii) Stationary points when $\frac{dy}{dx} = 0.$

That is, $\left(x^2 - 4x + 3\right) = 0$

$(x-3)(x-1) = 0.$

$\therefore\ x = 3$ or $x = 1.$

Stationary points are (3, 1) and (1, 5).

(iii) $\frac{d^2y}{dx^2} = 6x - 12$

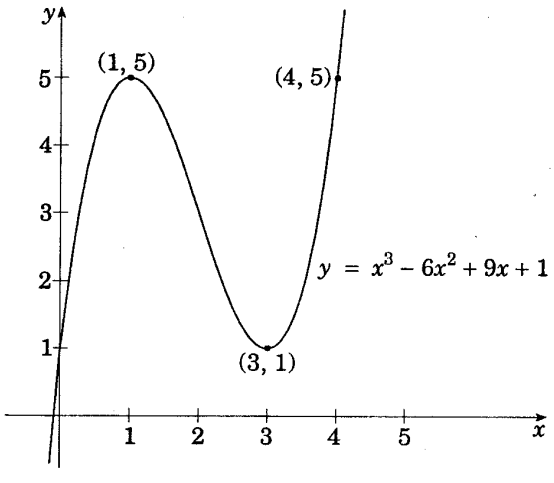
When $x = 1,\quad \frac{d^2y}{dx^2} = 6 - 12 < 0,$

$\therefore$ the curve is concave down and (1, 5) is a relative maximum.

When $x = 3,\quad \frac{d^2y}{dx^2} = 18 - 12 > 0,$

$\therefore$ the curve is concave up and (3, 1) is a relative minimum.

(iv)



(b) $\log_a 2 = x,\quad \log_a 3 = y$

$\log_a 12 = \log_a 4\times 3$

$= \log_a 4 + \log_a 3$

$= \log_a 2^2 + \log_a 3$

$= 2\log_a 2 + \log_a 3$

$= 2x + y.$

There are other possible answers (for example, $x\log_2 12$), but this is the simplest and most obvious.

(c) $\int_{-2}^{5} f(x)\,dx = 8 - 3 + 1$

$= 6.$

QUESTION 6

(a) $M = 10e^{-kt}$

(i) Initial mass occurs when $t = 0,$

that is, $M = 10e^{-k\times 0}$

$= 10.$

So the initial mass is 10 kg.

(ii) $t = 100,\ M = 5$

$M = 10e^{-kt}$

$\therefore\quad 5 = 10e^{-100k}$

$e^{-100k} = 0{\cdot}5$

$\ln e^{-100k} = \ln 0{\cdot}5$

$-100k = \ln 0{\cdot}5$

$k = \frac{\ln 0{\cdot}5}{-100}$

$= 6{\cdot}931\,471\,806 \times 10^{-3}$

$= 0{\cdot}006\,93$ (3 sig. figs).

(iii) $t = 1000,\ M = ?$

$M = 10e^{-k\times 1000}$

$= 0{\cdot}009\,765\,625$

$= 0{\cdot}009\,766$ kg (4 sig. figs)

$= 9{\cdot}766$ g.

OR Since the mass halves in 100 years, it will halve ten times in 1000 years.

$\therefore\quad$ Mass $= \frac{10}{2^{10}} = 0{\cdot}009\,766$ kg.

(iv) $t = ?,\ M = 8$

$8 = 10e^{-kt}$

$0{\cdot}8 = e^{-kt}$

$\ln 0{\cdot}8 = \ln e^{-kt}$

$\ln 0{\cdot}8 = -kt$

$t = \frac{\ln 0{\cdot}8}{-k}$

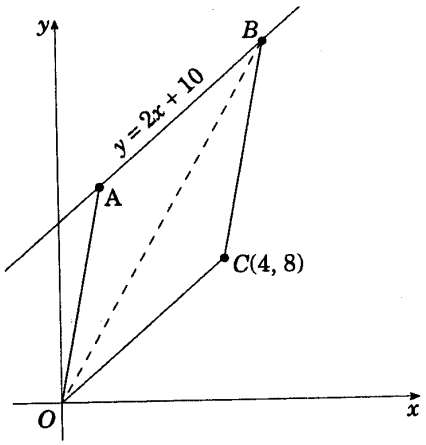
$= \frac{\ln 0{\cdot}8 \times 100}{\ln 0{\cdot}5}$

$= 32{\cdot}192\,809\,49$

$= 32{\cdot}2$ years (1 dec. pl.).

That is, it will take 32·2 years to reduce to 8 kg.

(b)



(i) Equation of AB is $y = 2x + 10$,
$\therefore$ gradient $= 2$.

Gradient $OC = \dfrac{8-0}{4-0} = 2$

$=$ gradient AB.

$\therefore OC \| AB$.

(ii) $\angle ABO = \angle BOC$ because they are alternate angles between parallel sides AB and OC.

(iii) Assume that in the congruent triangles ABO matches COB. With this assumption any of the following three arguments complete the proof.

$AO = BC$ (corresponding sides of congruent Δs)

$AB = CO$ (corresponding sides of congruent Δs)

$\therefore$ $OABC$ is a parallelogram (both pairs of opposite sides equal).

OR $\angle AOB = \angle CBO$ (corresponding angles of congruent Δs)

$OA \| CD$ (alternate angles equal)

$\therefore$ $OABC$ is a parallelogram (both pairs of opposite sides parallel).

OR $AB = CO$ (corresponding sides of congruent Δs)

$\therefore$ $OABC$ is a parallelogram (one pair of sides, AB and OC, are both parallel and equal).

There is a logic problem with this answer. The question does not actually say that ABO matches COB. Since there are no other restrictions on the points A and B, it is possible that the match could be done differently.

There are 5 other possibilities to consider. Two of them (ABO matches BCO and ABO matches OBC) are impossible, since the triangles would have to be equilateral.

The other three *are* possible. However, they all lead to isosceles triangles, from which one can deduce that a congruence with ABO matching COB is *also* possible. One can then complete the proof as above.

QUESTION 7

(a) (i) $\$A_1 = \$P \times 1{\cdot}08 - \$3000$

(ii) $\$A_2 = (\$P \times 1{\cdot}08 - \$3000)1{\cdot}08 - 3000$

$= \$P \times 1{\cdot}08^2 - \$3000 \times 1{\cdot}08 - 3000$

$= \$P \times 1{\cdot}08^2 - \$3000(1 + 1{\cdot}08)$.

(iii) $\$A_6 = \$P \times 1{\cdot}08^6 - \$3000\left(1 + 1{\cdot}08 + \cdots + 1{\cdot}08^5\right) = 0$.

$\therefore \quad \$P \times 1{\cdot}08^6 = \$3000\left(\dfrac{1{\cdot}08^6 - 1}{1{\cdot}08 - 1}\right)$

$$\$P = \frac{\$3000\left(1{\cdot}08^6 - 1\right)}{1{\cdot}08^6 \times 0{\cdot}08}$$

$= \$13\,868{\cdot}64$.

OR Investment $= P\left(1{\cdot}08^6\right)$.

Withdrawal

$= 3000 + 3000 \times 1{\cdot}08 + 3000 \times 1{\cdot}08^2 + \cdots + 3000 \times 1{\cdot}08^5$

$= 3000\left(1 + 1{\cdot}08 + 1{\cdot}08^2 + \cdots + 1{\cdot}08^5\right)$.

That is, GP where $a = 1$, $r = 1{\cdot}08$, $n = 6$,

$$S_6 = \frac{3000\left(1{\cdot}08^6 - 1\right)}{0{\cdot}08}$$

$= 22\,007{\cdot}79$.

Amount invested will be enough when investment = withdrawals.

That is, $P = \dfrac{2207{\cdot}79}{1{\cdot}08^6}$

$\therefore \quad \$P = \$13\,868{\cdot}64$.

(b) $x = 2\sin t - t, \quad t \geq 0$.

(i) $v = \dfrac{dx}{dt} = 2\cos t - 1$.

(ii) At $t = 0$, $\quad v = 2 - 1 = 1$,
$\therefore$ the particle is moving in a positive direction.

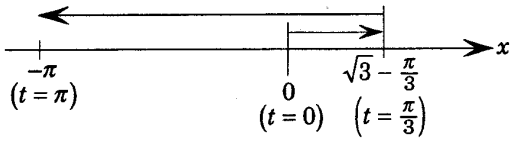
(iii) $2\cos t - 1 = 0$

$\cos t = \dfrac{1}{2}$

$t = \dfrac{\pi}{3}$ seconds.

(iv) $a = \dfrac{dv}{dt} = -2\sin t$.

Acceleration is negative when $0 < t < \pi$.



$$\text{Total distance} = \left(\sqrt{3} - \frac{\pi}{3}\right) + \left(\pi + \sqrt{3} - \frac{\pi}{3}\right)$$
$$= 2\sqrt{3} + \frac{\pi}{3}.$$

QUESTION 8

(a) (i) $y = e^{x^2}$

$x^2 = \log_e y$

Rotating about the y axis,

$$V = \pi\int_1^7 x^2\, dy$$
$$= \pi\int_1^7 \log_e y\, dy.$$

(ii)

y	1	4	7
$\log_e y$	0	1·386	1·946

(iii) $V = \pi\int_1^7 \log_e y\, dy$.

$\doteqdot \pi \times \frac{3}{2}[(0 + 1{\cdot}386) + (1{\cdot}386 + 1{\cdot}946)]$.

$\therefore$ Volume $\doteqdot$ 22·2 cubic units.

(b) (i) $\text{P(5 then 3)} = \text{P(5)} \times \text{P(3)}$
$= \frac{2}{5} \times \frac{2}{4}$
$= \frac{1}{5}$.

(ii) P(sum at least 8)
$= \text{P(sum 8)} + \text{P(sum 10)}$
$= \text{P(5, 3 or 3, 5)} + \text{P(5, 5)}$
$= \frac{1}{5} + \frac{1}{5} + \frac{2}{5} \times \frac{1}{4}$
$= \frac{1}{2}$.

(iii) The probability that the second card is a 3 is the same as the probability that the first card is a 3, that is, $\frac{2}{5}$.

OR $\text{P}(x, 3) = \text{P}(0, 3) + \text{P}(3, 3) + \text{P}(5, 3)$
$= \frac{1}{5} \times \frac{2}{4} + \frac{2}{5} \times \frac{1}{4} + \frac{2}{5} \times \frac{2}{4}$
$= \frac{2}{5}$.

(c) Stationary points occur when $f'(x) = 0$, that is, at $x = 1$ and at $x = 5$.

To find the minimum, use the first derivative or second derivative test.

First derivative test:

	$x < 1$	$x = 1$	$1 < x < 5$	$x = 5$	$x > 5$
$f'(x)$	> 0	0	< 0	0	> 0
$f(x)$	/	—	\	—	/

There is a local minimum at $x = 5$.

Second derivative test:

At $x = 1$, $f'(x)$ is decreasing so the curve is concave down (that is, $f''(x) < 0$). This is a local maximum.

At $x = 5$, $f'(x)$ is increasing so the curve is concave up (that is, $f''(x) > 0$). This is a local minimum.

QUESTION 9

(a) $A = \int_0^{\frac{\pi}{4}} \sec^2 x\, dx - A\Delta$
$= \left[\tan x\right]_0^{\frac{\pi}{4}} - \frac{1}{2} \times \left(\frac{\pi}{4}\right)^2$
$= \tan\frac{\pi}{4} - \tan 0 - \frac{\pi^2}{32}$
$= 1 - 0 - \frac{\pi^2}{32}$
$= 1 - \frac{\pi^2}{32}$
$= 0{\cdot}692$ (3 sig. figs).

(b) (i) 1. $\angle STR = \angle PTQ$ (vert. opp.)
2. $\angle SRT = \angle TPQ$ (alt. $\angle$s, $SR \| PQ$)
$\therefore \Delta SRT \| \| \Delta QPT$ (equal $\angle$s)
$\therefore \frac{SR}{PQ} = \frac{VT}{UT}$ (ratios of corresp. lengths in similar Δs. N.B. VT and UT are corresp. altitudes).

(ii) $\frac{SR}{4} = \frac{3 - y}{y}$

$SR = \frac{12 - 4y}{y}$

$\therefore SR = \frac{12}{y} - 4$.

(iii) $A(\Delta PTQ + \Delta RTS)$
$= \frac{1}{2}[SR \times (3 - y) + 4y]$
$= \frac{1}{2}\left[\left(\frac{12}{y} - 4\right)(3 - y) + 4y\right]$
$= \left(\frac{6}{y} - 2\right)(3 - y) + 2y$
$= \frac{18}{y} - 6 - 6 + 2y + 2y$
$= 4y - 12 + \frac{18}{y}$.

(iv) Minimum value of A occurs when $\frac{dA}{dy} = 0$ and $\frac{d^2A}{dy^2} > 0$.

$A = 4y - 12 + 18y^{-1}$

$\therefore \quad \frac{dA}{dy} = 4 - 18y^{-2}$

$\frac{d^2A}{dy^2} = 36y^{-3}$.

$\frac{dA}{dy} = 0, \quad \therefore \quad 4 - \frac{18}{y^2} = 0$

$4y^2 - 18 = 0$

$y^2 = \frac{18}{4}$

$y = \pm\frac{3\sqrt{2}}{2}$.

But $y > 0$, so $y = \frac{3\sqrt{2}}{2}$.

When $y = \frac{3\sqrt{2}}{2}$, $\frac{d^2A}{dy^2} = \frac{36}{\left(\frac{3\sqrt{2}}{2}\right)^3} > 0$.

$\therefore$ The value of y that minimises A is $\frac{3\sqrt{2}}{2}$ metres.

QUESTION 10

(a) (i) LHS: $\sin x = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

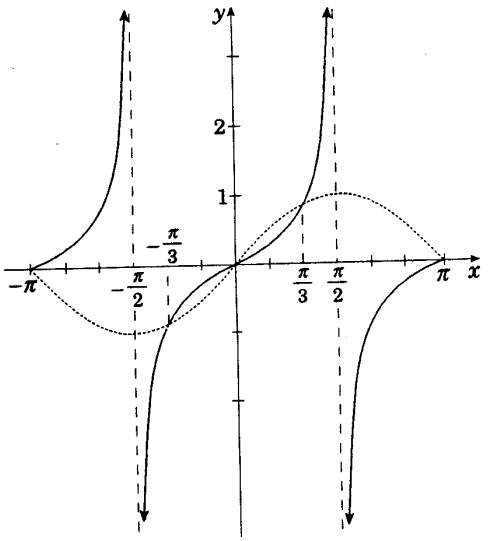
RHS: $\frac{1}{2}\tan x = \frac{1}{2}\tan\frac{\pi}{3}$

$= \frac{1}{2}\sqrt{3}$

$= \frac{\sqrt{3}}{2} =$ LHS.

That is, $\sin x = \frac{1}{2}\tan x$ when $x = \frac{\pi}{3}$.

(ii)

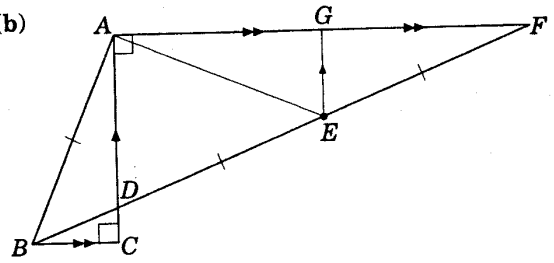


(iii) Solutions are $x = 0, \frac{\pi}{3}, -\frac{\pi}{3}$.

(iv) $\sin x \le \frac{1}{2}\tan x$ for $\frac{\pi}{3} < x \le \frac{\pi}{2}$

or $-\frac{\pi}{3} \le x \le 0$.

(b)



(i) $\angle BCD = \angle DAF$ (both 90°)

$\angle BDC = \angle ADF$ (vertically opposite $\angle$s)

$\therefore \quad \Delta DBC \| \Delta DFA$ (equal $\angle$s)

$\therefore \angle DBC = \angle DFA$ (corresponding $\angle$s in similar Δs).

OR $\angle BCD = \angle DAF$ (both 90°)

$\therefore \quad AF \| BC$ (alternate $\angle$s equal)

$\therefore \angle DBC = \angle DFA$ (alternate $\angle$s, $AF \| BC$).

(ii) $\angle AGE = 90°$ (coint. with $\angle GAD$, $AD \| GE$)

$\angle EGF = 90°$ ($\angle$ on a straight line)

$\therefore \angle AGE = \angle EGF$ (both 90°).

Since E is the midpoint of DF, G must be be the midpoint of AF (by equal intercept theorem, given that $GE \| AD$).

$\therefore \quad AG = GF$

$GE = GE$ (common)

$\therefore \Delta AGE \equiv \Delta FGE$ (SAS).

(iii) Let $\angle DBC = \alpha$

$\angle AFE = \alpha$ (from (i))

$\angle EAF = \angle AFE$ (corresp. $\angle$s, $\Delta AGE \equiv \Delta FGE$)

$= \alpha$,

$\therefore \quad \angle AED = 2\alpha$ (ext. $\angle$ of Δ equals sum of int. opp. $\angle$s).

Now $AE = EF$ (corresp. sides, $\Delta AGE \equiv \Delta FGE$)

$\therefore \quad AB = AE$ (both equal to EF)

$\therefore \quad \angle ABD = \angle AED$ (base $\angle$s of isos. Δ)

$= 2\alpha$

$= 2\angle DBC$.

END OF 2/3 UNIT (COMMON) MATHEMATICS SOLUTIONS