1998 HIGHER SCHOOL CERTIFICATE SOLUTIONS

2/3 UNIT (COMMON) MATHEMATICS

QUESTION 1

(a) $\frac{3}{11} = 0\dot{2}\dot{7}$.

(b) $|-5|-|8| = 5-8 = -3$.

(c) $P(3 \text{ heads}) = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{8}$.

(d) $\int x^2+7\,dx = \frac{1}{3}x^3+7x+C$.

(The '+ C' is optional. Any particular value of C is also correct.)

(e)
$$\sin\left(\frac{\pi}{4}\right)+\sin\left(\frac{2\pi}{3}\right) = \frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{2}$$
$$= \frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}$$
$$= \frac{\sqrt{2}+\sqrt{3}}{2}.$$

(f)
$$\frac{1}{3-\sqrt{2}}+\frac{1}{3+\sqrt{2}} = \frac{3+\sqrt{2}}{9-2}+\frac{3-\sqrt{2}}{9-2}$$
$$= \frac{6}{7}.$$

(g) 137·5% of wholesale price = \$3·08.

1% of wholesale price = \$3·08 ÷ 137·5
= \$0·0224.

Wholesale price (100%) = \$0·0224 × 100
= \$2·24.

QUESTION 2

(a) (i) $\frac{d}{dx}\left(3x^2+4\right)^5$

$= 5\left(3x^2+4\right)^4.(6x)$, using chain rule,

$= 30x\left(3x^2+4\right)^4$.

(ii) $\frac{d}{dx}x\sin(x+1) = x.\cos(x+1)+\sin(x+1)$, using product rule.

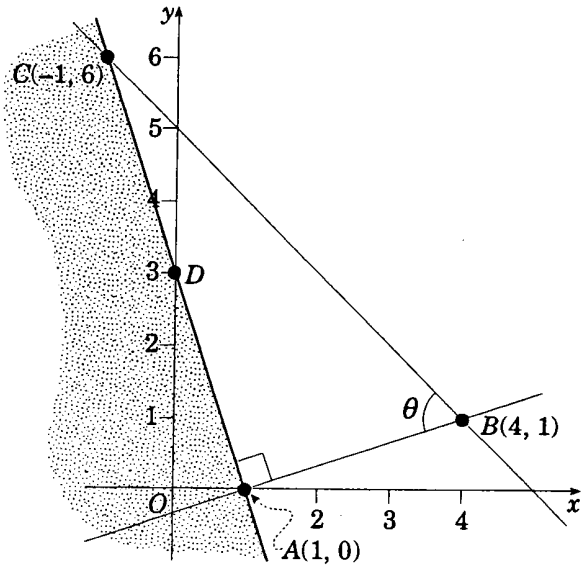
(iii) $\frac{d}{dx}\left(\frac{\tan x}{x}\right) = \frac{x\sec^2 x-\tan x}{x^2}$, using quotient rule.

(b) (i)
$$\int_1^2 \frac{1}{x^2}\,dx = \int_1^2 x^{-2}\,dx$$
$$= \left[\frac{1}{-1}x^{-1}\right]_1^2$$
$$= \left[-\frac{1}{x}\right]_1^2$$
$$= -\frac{1}{2}-\left(-\frac{1}{1}\right)$$
$$= \frac{1}{2}.$$

(ii)
$$\int_0^3 e^{4x}\,dx = \left[\frac{1}{4}e^{4x}\right]_0^3$$
$$= \frac{1}{4}e^{12}-\frac{1}{4}e^0$$
$$= \frac{1}{4}e^{12}-\frac{1}{4}$$
$$= \frac{1}{4}\left(e^{12}-1\right).$$

(c)
$$\int \frac{x}{x^2+3}\,dx = \frac{1}{2}\int \frac{2x}{x^2+3}\,dx$$
$$= \frac{1}{2}\ln\left(x^2+3\right)+C$$
$$\left(\text{or } \ln\sqrt{x^2+3}+C\right).$$

QUESTION 3



(a) Substitute $A(1, 0)$ into $3x+y = 3$.

$$\begin{aligned}\text{LHS} &= 3(1)+0\\ &= 3\\ &= \text{RHS.}\end{aligned}$$

$\therefore$ A lies on the line.

Substitute $C(-1, 6)$ into $3x+y = 3$.

$$\begin{aligned}\text{LHS} &= 3(-1)+6\\ &= -3+6\\ &= 3\\ &= \text{RHS.}\end{aligned}$$

$\therefore$ C lies on the line.

OR Finding equation of line which passes through $A(1, 0)$ and $C(-1, 6)$,

$$m_{AC} = \frac{6-0}{-1-1} = \frac{6}{-2} = -3$$

$$y-0 = -3(x-1)$$

$$y = -3x+3$$

$3x+y = 3$ is the line:

(b) $m_{AB} = \dfrac{1-0}{4-1} = \dfrac{1}{3}$.

(c) $AB = \sqrt{(4-1)^2+(1-0)^2}$

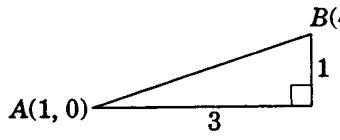
$= \sqrt{3^2+1^2}$

$= \sqrt{9+1}$

$= \sqrt{10}$ units.

OR by Pythagoras' theorem: $AB = \sqrt{3^2+1^2}$

$= \sqrt{10}$ units.



(d) AB and AC are perpendicular if

$$m_{AB} \times m_{AC} = -1.$$

$m_{AB} = \frac{1}{3}$ from (b).

$$m_{AC} = \frac{6-0}{-1-1} = \frac{6}{-2} = -3.$$

OR AC has equation $3x+y = 3$

$y = -3x+3$

$\therefore$ $m_{AC} = -3$.

$$m_{AB} \cdot m_{AC} = \tfrac{1}{3}(-3) = -1$$

$\therefore$ $AB \perp AC$.

(e) ΔCAB is right angled at $\angle A$, from (d).

$$\therefore \quad \tan\theta = \frac{CA}{AB} = \frac{CA}{\sqrt{10}}$$

$$CA = \sqrt{(-1-1)^2+(6-0)^2}$$

$$= \sqrt{4+36}$$

$$= \sqrt{40}$$

$$= 2\sqrt{10}$$

$$\therefore \quad \tan\theta = \frac{2\sqrt{10}}{\sqrt{10}} = 2.$$

(f) Circle, centre $A(1, 0)$, radius $AB = \sqrt{10}$ has equation

$$(x-1)^2+(y-0)^2 = \left(\sqrt{10}\right)^2$$

$$(x-1)^2+y^2 = 10.$$

(g) Method 1: [follows from (f)]

If $AD = AB = \sqrt{10}$, then D also lies on the circle $(x-1)^2+y^2 = 10$.

Solve simultaneously:

$$3x+y = 3 \qquad \text{—①}$$

$$(x-1)^2+y^2 = 10 \qquad \text{—②}$$

From ①: $y = 3-3x$.

Substitute into ②:

$$(x-1)^2+(3-3x)^2 = 10$$

$$x^2-2x+1+9-18x+9x^2 = 10$$

$$10x^2-20x = 0$$

$$10x(x-2) = 0$$

$$x = 0 \text{ or } 2.$$

But D lies between A and C,

$\therefore$ $x = 0$.

Substitute $x = 0$ into ①: $0+y = 3$

$y = 3$.

D has coordinates $(0, 3)$.

Method 2:

$CA = 2\sqrt{10}$ from (e).

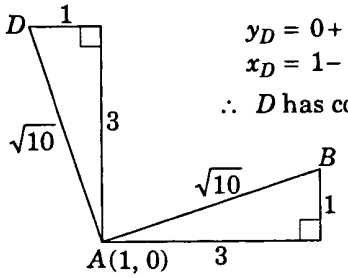
$AB = \sqrt{10}$ from (c).

$\therefore$ $AD = \sqrt{10}$ and D is the midpoint of AC.

$A(1, 0)$, $C(-1, 6)$.

D has coordinates $\left(\dfrac{1+-1}{2}, \dfrac{0+6}{2}\right) = (0, 3)$.

Method 3: [follows from (d)]



$y_D = 0+3 = 3$.

$x_D = 1-1 = 0$.

$\therefore$ D has coordinates $(0, 3)$.

Note to students:

Always look for connections between parts of a question, even if the instruction 'hence, prove' is not explicitly stated.

(h) Test (0, 0). LHS $= 3(0)+0 = 0$
≤ 3.

$\therefore$ Region is half-plane which includes (0, 0). (See diagram.)

QUESTION 4

(a) $h = 1$

x	1·0	2·0	3·0
$f(x)$	1·7	9·0	4·3

(i) Simpson's rule:

$$\int_1^3 f(x)\,dx \doteqdot \frac{h}{3}[f(1)+4f(2)+f(3)]$$
$$= \tfrac{1}{3}[1{\cdot}7+4\times 9{\cdot}0+4{\cdot}3]$$
$$= 14.$$

(ii) Trapezoidal rule:

$$\int_1^3 f(x)\,dx \doteqdot \frac{h}{2}[f(1)+2f(2)+f(3)]$$
$$= \tfrac{1}{2}[1{\cdot}7+2\times 9{\cdot}0+4{\cdot}3]$$
$$= 12.$$

(b) $_$, $_$, 32, $_$, $_$, 17

(i) $T_3 = a+2d = 32$ —①
$T_6 = a+5d = 17$. —②

②−①: $3d = -15$.
Common difference $d = -5$.

(ii) $S_{10} = \frac{10}{2}(2a+9d)$

Using ① to find a:
$a+2d = 32$
$a+2(-5) = 32$
$a-10 = 32$
$a = 42$.

$\therefore$ $S_{10} = \frac{10}{2}[2{\cdot}42+9(-5)]$
$= 5(39)$
$= 195$.

(c) 16, $_$, $_$, $\frac{1}{4}$

(i) $T_1 = a = 16$. —①
$T_4 = ar^3 = \frac{1}{4}$. —②

Substitute ① into ②: $16r^3 = \frac{1}{4}$
$r^3 = \frac{1}{64}$.

Common ratio $r = \frac{1}{4}$.

(ii) $S_\infty = \dfrac{a}{1-r} = \dfrac{16}{1-\frac{1}{4}} = 21\frac{1}{3}$.

Note: The constant multiplication feature of calculators can be used to sum any geometric series.

For the above limiting sum, press (on Casio calculators):

$\frac{1}{4}$ [×] [×] 16 [Min] [M+] [M+] [M+] ⋯
[M+] [MR].

Repeat [M+] until the display shows a negligibly small number.

(d) (i) $x^2 = 8(y+3)$ is in the form
$(x-h)^2 = 4a(y-k)$,
where $(h, k) = (0, -3)$ is the vertex and $a = 2$ is the focal length,
$\therefore$ vertex is $(0, -3)$.

OR $\dfrac{x^2}{8} = y+3$

$y = \dfrac{x^2}{8} - 3$.

Vertex at $x = \dfrac{-b}{2a} = \dfrac{-0}{2\left(\frac{1}{8}\right)} = 0$.

When $x = 0$, $y = \dfrac{0}{8} - 3 = -3$.

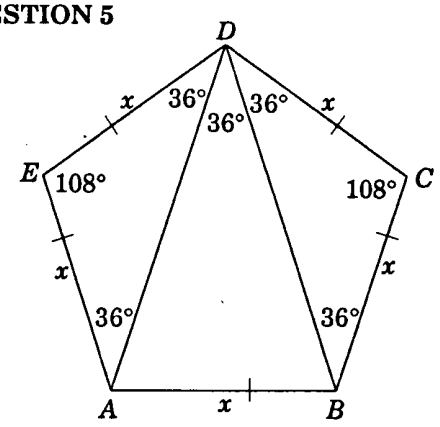
$\therefore$ Vertex is $(0, -3)$.

(ii) Directrix is horizontal line a (= 2) units below $(0, -3)$.

$\therefore$ Directrix is $y = -5$.

QUESTION 5

(a)



(i) $CD = BC$ (both x, given)
$\therefore$ ΔBCD is isosceles.
$\therefore$ $\angle CBD = \dfrac{180°-108°}{2} = 36°$.

(ii) In ΔBCD and ΔDEA,
$CD = ED$ (given)
$CB = EA$ (given)
$\angle C = \angle E = 108°$ (given)
$\therefore$ $\Delta BCD \equiv \Delta DEA$ (SAS).

(iii) $\angle ADB = \angle EDC - \angle EDA - \angle CDB$
$= 108° - 36° - 36°$
$= 36°$.

(iv) Area of the pentagon $DCBAE$
$= \text{area } \Delta BCD + \text{area } \Delta DEA + \text{area } \Delta DBA$
$= 2 \times \text{area } \Delta BCD + \text{area } \Delta DBA$
$(\Delta BCD \equiv \Delta DEA)$.

Area ΔBCD
$= \frac{1}{2}x.x.\sin 108° \left(\text{using } A = \frac{1}{2}ab\sin C\right)$
$= \frac{1}{2}x^2 \sin 108°$.

$\therefore$ Area of pentagon
$= x^2 \sin 108° + \text{area } \Delta DBA$.

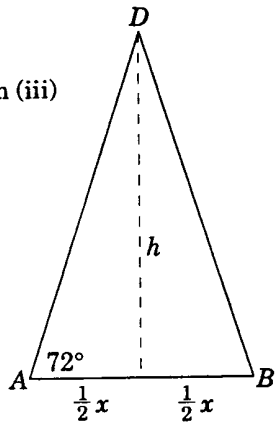
We present two methods for finding the area of ΔDBA.

Method 1:

$\angle DAB = \frac{180° - 36°}{2}$ from (iii)
$= 72°$.

$\tan 72° = \frac{h}{\frac{1}{2}x}$

$\therefore \quad h = \frac{1}{2}x \tan 72°$.



Area ΔDBA
$= \frac{1}{2} \times \text{base} \times \text{height}$
$= \frac{1}{2} \times x \times \frac{1}{2}x \tan 72°$
$= \frac{1}{4}x^2 \tan 72°$.

Using this method:
Area of pentagon $= x^2 \sin 108° + \frac{1}{4}x^2 \tan 72°$.

Method 2:

Area ΔDBA
$= \frac{1}{2}(DB)(DA)\sin 36°$
$= \frac{1}{2}(DB)^2 \sin 36°$ $(DA = DB,\ \Delta BCD \equiv \Delta DEA)$.

Using the cosine rule on ΔBCD,
$DB^2 = x^2 + x^2 - 2x.x.\cos 108°$
$= 2x^2 - 2x^2 \cos 108°$
$= 2x^2(1 - \cos 108°)$.

Area $\Delta DBA = \frac{1}{2} \times 2x^2(1 - \cos 108°)\sin 36°$
$= x^2(1 - \cos 108°)\sin 36°$.

Area of pentagon
$= x^2 \sin 108° + x^2(1 - \cos 108°)\sin 36°$.

(There are many other correct expressions for the area of the pentagon. For example, DB could have been found using the sine rule. Also, the identities $\sin 108° \equiv \sin 72°$ and $\cos 108° \equiv -\cos 72°$ could have been used.

If your answer is different, substitute $x = 2$ into your expression. If you get 6·881 909 602, then you are probably correct.)

(b) (i) When $t = 0$, $P = 1\,000\,000$
$1\,000\,000 = Ae^0$
$A = 1\,000\,000$.

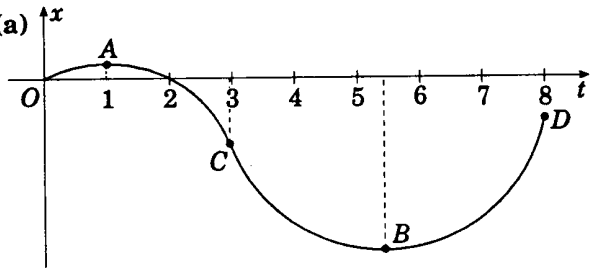
(ii) $\therefore \quad P = 1\,000\,000e^{kt}$
When $t = 2$, $P = 1\,072\,500$
$1\,072\,500 = 1\,000\,000e^{2k}$
$1{\cdot}0725 = e^{2k}$
$2k = \ln(1{\cdot}0725)$
$k = \frac{1}{2}\ln(1{\cdot}0725)$
$(\doteqdot 0{\cdot}034\,996)$.

(iii) Substitute $P = 2\,000\,000$.
$2\,000\,000 = 1\,000\,000e^{kt}$
$2 = e^{kt}$
$\ln 2 = kt$
$t = \frac{\ln 2}{k}$
$= \frac{\ln 2}{\frac{1}{2}\ln(1{\cdot}0725)}$
$\doteqdot 19{\cdot}806$ years.

The population will reach 2 000 000 after 19·8 years.

QUESTION 6

(a)



(i) $v = x'(t) = 0$ at stationary points of curve, i.e. at $t = 1$ and $t = 5{\cdot}4$ seconds (A and B on the graph).

(ii) $a = x''(t) = 0$ at points of inflection of the curve, i.e. at $t = 3$ seconds, where the curvature changes from concave down to concave up (C on the graph).

(iii) Distance from S is greatest at $t = 5$ seconds (B on the graph, where distance from t axis is greatest).

(iv) Velocity is greatest when the gradient of the curve is steepest, i.e. at $t = 8$ (D on the graph).

(b) (i) Stationary point at $f'(x) = 0$
$f'(x) = e^{-2x} - 2xe^{-2x} = 0$
$e^{-2x}(1 - 2x) = 0$
$1 - 2x = 0 \quad (e^{-2x} \neq 0)$
$x = \frac{1}{2}$.

(ii) $f(x)$ is increasing when $f'(x) > 0$.

$$f'(x) = e^{-2x}(1-2x) > 0$$
$$1-2x > 0 \quad (e^{-2x} > 0)$$
$$1 > 2x$$
$$x < \tfrac{1}{2}.$$

(iii) $f(x)$ has a point of inflection if $f''(x) = 0$ and $f''(x)$ changes sign.

$$f''(x) = 4xe^{-2x} - 4e^{-2x} = 0$$
$$4e^{-2x}(x-1) = 0$$
$$x - 1 = 0 \quad (4e^{-2x} \neq 0)$$
$$x = 1.$$

$f''(x) = 4e^{-2x}(x-1)$.

$4e^{-2x}$ is always positive,

so when $x < 1$, $f''(x) < 0$

$x > 1$, $f''(x) > 0$.

OR

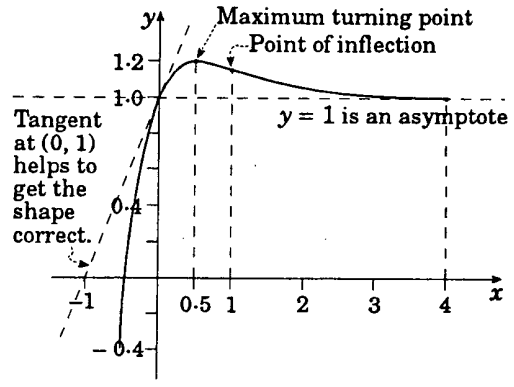
x	0	1	2
$f''(x)$	−4	0	0·07
	−		+

$\therefore$ Concavity changes from negative (concave down) to positive (concave up).

$\therefore$ Point of inflection occurs at $x = 1$, and graph is concave up for $x > 1$.

(iv)

x	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	4
$f(x)$	−0·359	1	1·184	1·135	1·001
		$f'(0) = 1$	Stationary point, from (i)	Point of inflection, from (iii)	
	Increasing, from (ii)		Decreasing		
	Concave down			Concave up, from (iii)	



(v) As $x \to \infty$, $xe^{-2x} \to 0$,

$\therefore$ $f(x) = xe^{-2x} + 1 \to 1$.

$\therefore$ The graph approaches the line $y = 1$.

QUESTION 7

(a) (i) $\Delta = b^2 - 4ac$
$= 2^2 - 4.3.k$
$= 4 - 12k.$

(ii) Real roots if $\Delta \geq 0$.
$$4 - 12k \geq 0$$
$$4 \geq 12k$$
$$k \leq \tfrac{4}{12}$$
$$k \leq \tfrac{1}{3}.$$

(b) (i) $y = 1 + \sqrt{3}\sin x + \cos x.$

$$\frac{dy}{dx} = \sqrt{3}\cos x - \sin x.$$

At $x = \frac{5\pi}{6}$,

$$y = 1 + \sqrt{3}\sin\frac{5\pi}{6} + \cos\frac{5\pi}{6}$$
$$= 1 + \sqrt{3}.\frac{1}{2} + \frac{-\sqrt{3}}{2}$$
$$= 1.$$

$\left(\frac{5\pi}{6}, 1\right)$

$$\frac{dy}{dx} = \sqrt{3}\cos\frac{5\pi}{6} - \sin\frac{5\pi}{6}$$
$$= \sqrt{3}.\frac{-\sqrt{3}}{2} - \frac{1}{2}$$
$$= \frac{-3}{2} - \frac{1}{2}$$
$$= -2.$$

$\therefore$ Equation of tangent is

$$y - 1 = -2\left(x - \frac{5\pi}{6}\right)$$
$$y - 1 = -2x + \frac{5\pi}{3}$$
$$y = -2x + \frac{5\pi}{3} + 1,$$

or in general form:

$$3y = -6x + 5\pi + 3$$
$$6x + 3y - 5\pi - 3 = 0.$$

(ii) Max/min values at $\frac{dy}{dx} = 0$:

$$\sqrt{3}\cos x - \sin x = 0$$
$$\sqrt{3}\cos x = \sin x$$
$$\sqrt{3} = \tan x$$

$\therefore$ $x = \frac{\pi}{3}, \frac{4\pi}{3}$.

$$\frac{d^2y}{dx^2} = -\sqrt{3}\sin x - \cos x.$$

When $x = \frac{\pi}{3}$,

$$\frac{d^2y}{dx^2} = -\sqrt{3}\sin\frac{\pi}{3} - \cos\frac{\pi}{3}$$

$$= -\sqrt{3}\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{2}$$
$$= -2$$
$$< 0.$$

$\therefore$ $x = \frac{\pi}{3}$ gives a maximum value.

When $x = \frac{\pi}{3}$,

$$y = 1 + \sqrt{3}\sin\frac{\pi}{3} + \cos\frac{\pi}{3}$$
$$= 1 + \sqrt{3}\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}$$
$$= 3.$$

$\therefore$ The maximum value is 3.

When $x = \frac{4\pi}{3}$,

$$\frac{d^2y}{dx^2} = -\sqrt{3}\sin\frac{4\pi}{3} - \cos\frac{4\pi}{3}$$
$$= -\sqrt{3}\left(-\frac{\sqrt{3}}{2}\right) - \left(-\frac{1}{2}\right)$$
$$= 2$$
$$> 0.$$

$\therefore$ $x = \frac{4\pi}{3}$ gives a minimum value.

When $x = \frac{4\pi}{3}$,

$$y = 1 + \sqrt{3}\sin\frac{4\pi}{3} + \cos\frac{4\pi}{3}$$
$$= 1 + \sqrt{3}\left(-\frac{\sqrt{3}}{2}\right) - \frac{1}{2}$$
$$= -1.$$

$\therefore$ The minimum value is -1.

(c) (i) Perimeter $= r + r + r\theta = 8$

$$2r + r\theta = 8$$
$$r(2+\theta) = 8$$
$$r = \frac{8}{2+\theta}.$$

(ii) $A = \frac{1}{2}r^2\theta$

$$= \frac{1}{2}\left(\frac{8}{2+\theta}\right)^2\theta$$
$$= \frac{1}{2}\cdot\frac{64}{(2+\theta)^2}\cdot\theta$$
$$= \frac{32\theta}{(\theta+2)^2}.$$

(iii) Maximum area occurs at $\frac{dA}{d\theta} = 0$.

$$A = \frac{32\theta}{(\theta+2)^2}$$

Using quotient rule with

$u = 32\theta$ $\quad v = (\theta+2)^2$

$\frac{du}{d\theta} = 32$ $\quad \frac{dv}{d\theta} = 2(\theta+2)$.

$$\frac{dA}{d\theta} = \frac{32(\theta+2)^2 - 64\theta(\theta+2)}{(\theta+2)^4} = 0$$

$$32(\theta+2)^2 - 64\theta(\theta+2) = 0 \quad (\theta \neq -2)$$
$$32(\theta+2) - 64\theta = 0 \quad (\theta \neq -2)$$
$$32\theta + 64 - 64\theta = 0$$
$$64 - 32\theta = 0$$
$$64 = 32\theta$$
$$\theta = 2.$$

But this value $\theta = 2$ lies outside of the domain $0 \le \theta \le \frac{\pi}{2}$. Test endpoints of $0 \le \theta \le \frac{\pi}{2}$ to find maximum area.

When $\theta = 0$, $\quad A = 0$

$\theta = \frac{\pi}{2}$, $\quad A = \frac{32\frac{\pi}{2}}{\left(\frac{\pi}{2}+2\right)^2}$

$$= \frac{64\pi}{(\pi+4)^2}$$
$$= 3{\cdot}9422 \text{ cm}^2.$$

$\therefore$ Maximum area is $3{\cdot}9422$ cm^2 at $\theta = \frac{\pi}{2}$.

QUESTION 8

(a) (i) $R = 100t - t^3$.

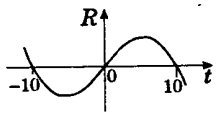
When $t = 8$, $\quad R = 100(8) - (8)^3$

$$= 288 \text{ kg/s}.$$

(ii) $R \ge 0$ since you can't untip the sand, so we examine values of t for which $100t - t^3 \ge 0$.

$$100t - t^3 = t(100 - t^2)$$
$$= t(10+t)(10-t).$$

By testing values and/or noting that R is a cubic function, we find that $100t - t^3 \ge 0$ when $t \le -10$ or when $0 \le t \le 10$.



In this problem $t \ge 0$, so we are left with $0 \le t \le 10$.

OR $\quad 100t - t^3 \ge 0$

$$100t \ge t^3$$
$$100 \ge t^2 \quad (\text{since } t \ge 0)$$
$$0 \le t \le 10 \quad (\text{again since } t \ge 0)$$

$\therefore$ The maximum value of T is $T = 10$.

(iii) Maximum rate at $\frac{dR}{dt} = 0$.

$$\frac{dR}{dt} = 100 - 3t^2 = 0$$
$$3t^2 = 100$$
$$t^2 = \frac{100}{3}$$
$$t = \frac{10}{\sqrt{3}} \quad (t > 0).$$

We know this is a maximum because R is a continuous function of t, and $R \geq 0$ on $0 \leq t \leq 10$, $R = 0$ at the endpoints, and this is the only stationary point on this domain.

When $t = \frac{10}{\sqrt{3}}$, $R = 100\left(\frac{10}{\sqrt{3}}\right) - \left(\frac{10}{\sqrt{3}}\right)^3$

$= \frac{1000}{\sqrt{3}} - \frac{1000}{3\sqrt{3}}$

$= \frac{1000}{\sqrt{3}}\left(\frac{2}{3}\right)$

$= \frac{2000}{3\sqrt{3}}$

$= \frac{2000\sqrt{3}}{9}$ kg/s

$\left(\doteqdot 384{\cdot}90 \text{ kg/s}\right)$.

(iv) If A is the amount of sand at time t, then

$R = \frac{dA}{dt} = 100t - t^3$

$A = 50t^2 - \frac{1}{4}t^4 + C.$

When $t = 0$, $A = 300$

$300 = 0 - 0 + C$

$\therefore \quad C = 300$

$\therefore \quad A = 50t^2 - \frac{1}{4}t^4 + 300.$

(v) When $t = 8$, $A = 50(8)^2 - \frac{1}{4}(8)^4 + 300 = 2476.$

$\therefore$ Weight of sand tipped

$= 2476 - 300$ (original sand)

$= 2176$ kg.

OR Weight of sand tipped

$= \int_0^8 100t - t^3 \, dt$

$= \left[50t^2 - \frac{1}{4}t^4\right]_0^8$

$= \left[50\left(8^2\right) - \frac{1}{4}\left(8^4\right)\right] - [0 - 0]$

$= 2176$ kg.

(b) (i) $y = \log_2 x = \frac{\ln x}{\ln 2}$ (change of base law)

$y \ln 2 = \ln x$

$x = e^{y \ln 2}.$

$V = \pi \int_0^3 x^2 \, dy$

$= \pi \int_0^3 \left(e^{y \ln 2}\right)^2 dy$

$= \pi \int_0^3 \left(e^{2y \ln 2}\right) dy$

$= \pi \int_0^3 e^{y \ln 2^2} \, dy$

$= \pi \int_0^3 e^{y \ln 4} \, dy.$

(ii) $V = \pi \int_0^3 e^{y \ln 4} \, dy$

$= \pi \left[\frac{1}{\ln 4} e^{y \ln 4}\right]_0^3$

$= \pi \left[\frac{1}{\ln 4} e^{3 \ln 4} - \frac{1}{\ln 4} e^0\right]$

$= \frac{\pi}{\ln 4}\left[e^{3 \ln 4} - 1\right]$

$= \frac{\pi}{\ln 4}\left[\left(e^{\ln 4}\right)^3 - 1\right]$

$= \frac{\pi}{\ln 4}\left(4^3 - 1\right)$

$= \frac{63\pi}{\ln 4}$ units^3

$\left(\doteqdot 142{\cdot}77 \text{ units}^3\right)$.

QUESTION 9

(a) $\ln(7x - 12) = 2 \ln x$

$= \ln x^2.$

$\therefore \quad 7x - 12 = x^2$

$0 = x^2 - 7x + 12$

$0 = (x - 3)(x - 4)$

$\therefore \; x = 3$ or 4.

Note: $\ln(7x - 12)$ is defined only for

$7x - 12 > 0$

$x > \frac{12}{7}$

$x > 1\frac{5}{7}$

and both solutions $x = 3$, 4 fall in this range.

(b) Area $= \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \cos 2x - \sin x \, dx$ [This is only possible because the 2 graphs do not cross in the domain $-\frac{\pi}{2} < x < \frac{\pi}{6}$.]

$= \left[\frac{1}{2}\sin 2x + \cos x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{6}}$

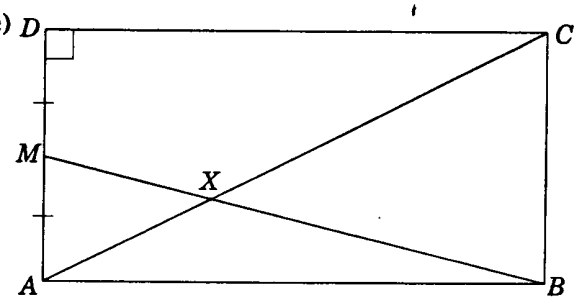
$= \left(\frac{1}{2}\sin\frac{\pi}{3} + \cos\frac{\pi}{6}\right) - \left[\frac{1}{2}\sin(-\pi) + \cos\left(-\frac{\pi}{2}\right)\right]$

$= \left(\frac{1}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) - (0 + 0)$

$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}$

$= \frac{3\sqrt{3}}{4}$ units^2.

(c)



(i) In ΔAXM and ΔBXC,

$\angle MXA = \angle CXB$ (vert. opp. angles)

$\angle MAX = \angle XCB$ (alt. angles between $DA \parallel CB$)

(or $\angle AMX = \angle CBX$)

$\therefore\ \Delta AXM \parallel\!\parallel \Delta BXC$ (They have the same $\angle$s).

(ii) $MA = \frac{1}{2}AD$ (M is midpoint of AD)

$= \frac{1}{2}CB$ ($AD = CB$, $ABCD$ is a rectangle)

$\therefore\ \dfrac{MA}{CB} = \frac{1}{2}$.

MA and CB are corresponding sides in the similar triangles, so this is the scale ratio.

$\therefore\ \dfrac{AX}{CX} = \frac{1}{2}$ (AX, CX are corresponding sides)

$\therefore\ \dfrac{AX}{CX} + \dfrac{CX}{CX} = \frac{1}{2} + 1$

i.e. $\dfrac{AX + CX}{CX} = \frac{3}{2}$

$\therefore\ \dfrac{AC}{CX} = \frac{3}{2}$

$\therefore\ 3CX = 2AC.$

(iii) Squaring both sides of (ii):

$(3CX)^2 = (2AC)^2$

$9(CX)^2 = 4(AC)^2.$

But $(AC)^2$

$= (AB)^2 + (CB)^2$ (by Pythagoras' theorem)

$= (AB)^2 + (DA)^2$

$= (AB)^2 + \left(\frac{1}{2}AB\right)^2$ (given $AB = 2AD$)

$= (AB)^2 + \frac{1}{4}(AB)^2$

$= \frac{5}{4}(AB)^2.$

$\therefore\ 9(CX)^2 = 4 \cdot \frac{5}{4}(AB)^2$

$= 5(AB)^2.$

QUESTION 10

(a) (i)

Pink die \ Blue die	4	6	8	9	10	12
2						
3						
5	W					
7	W	W				
11	W	W	W	W	W	
13	W	W	W	W	W	W

$\text{P(win)} = \dfrac{14}{36} = \dfrac{7}{18}.$

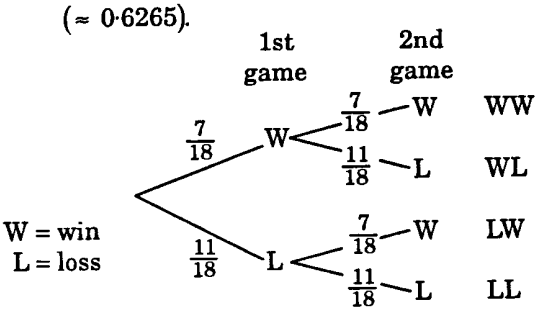
(ii) P(win at least once)

$= 1 - \text{P(LL)}$ (see tree diagram)

$= 1 - \dfrac{11}{18} \cdot \dfrac{11}{18}$

$= \dfrac{203}{324}$

($\approx 0{\cdot}6265$).



OR P(win at least once)

$= \text{P(WW)} + \text{P(WL)} + \text{P(LW)}$

$= \dfrac{7}{18} \cdot \dfrac{7}{18} + \dfrac{7}{18} \cdot \dfrac{11}{18} + \dfrac{11}{18} \cdot \dfrac{7}{18}$

$= \dfrac{49}{324} + \dfrac{77}{324} + \dfrac{77}{324}$

$= \dfrac{203}{324}.$

(b) (i) No. of fish after 1st harvest (Dec. '98)

$= 100\,000 + (10\% \times 100\,000) - 15\,400$

$= 100\,000(1{\cdot}1) - 15\,400$

$= 94\,600.$

No. of fish after 2nd harvest (Dec. '99)

$= 94\,600(1{\cdot}1) - 15\,400$

$= 88\,660.$

(ii) Noticing the pattern in (i):

$F_1 = 100\,000(1{\cdot}1) - 15\,400.$

$F_2 = F_1(1{\cdot}1) - 15\,400$

$= \left[100\,000(1{\cdot}1) - 15\,400\right](1{\cdot}1) - 15\,400$

$= 100\,000(1{\cdot}1)^2 - 15\,400(1{\cdot}1) - 15\,400$

$= 100\,000(1{\cdot}1)^2 - 15\,400(1{\cdot}1 + 1).$

$F_3 = F_2(1{\cdot}1) - 15\,400$

$= \left[100\,000(1{\cdot}1)^2 - 15\,400(1{\cdot}1 + 1)\right](1{\cdot}1) - 15\,400$

$= 100\,000(1{\cdot}1)^3 - 15\,400\left(1{\cdot}1^2 + 1{\cdot}1\right) - 15\,400$

$= 100\,000(1{\cdot}1)^3 - 15\,400\left(1{\cdot}1^2 + 1{\cdot}1 + 1\right).$

Generally,

$F_n = 100\,000(1{\cdot}1)^n - 15\,400\left(1{\cdot}1^{n-1} + \cdots + 1{\cdot}1^2 + 1{\cdot}1 + 1\right)$

↑ Geometric series $a = 1$, $r = 1{\cdot}1$, $n = n$

$S_n = \dfrac{1\left(1 - 1{\cdot}1^n\right)}{1 - 1{\cdot}1} = \dfrac{1 - 1{\cdot}1^n}{-0{\cdot}1}.$

$$\therefore \quad F_n = 100\,000(1{\cdot}1)^n - 15\,400\left(\frac{1-1{\cdot}1^n}{-0{\cdot}1}\right)$$
$$= 100\,000(1{\cdot}1)^n + 154\,000\left(1-1{\cdot}1^n\right)$$
$$= 100\,000(1{\cdot}1)^n + 154\,000 - 154\,000(1{\cdot}1)^n$$
$$= 154\,000 - 54\,000(1{\cdot}1)^n.$$

(iii) The farmer sells all his fish when $F_n = 0$.

$$F_n = 154\,000 - 54\,000(1{\cdot}1)^n = 0$$
$$\therefore \quad 54\,000(1{\cdot}1)^n = 154\,000$$
$$(1{\cdot}1)^n = \frac{77}{27}$$
$$\log(1{\cdot}1)^n = \log\left(\frac{77}{27}\right)$$
$$n\log(1{\cdot}1) = \log\left(\frac{77}{27}\right)$$
$$n = \frac{\log\left(\frac{77}{27}\right)}{\log(1{\cdot}1)}$$
$$\doteqdot 10{\cdot}9953 \text{ years}$$
$$\doteqdot 11 \text{ years (11th harvest).}$$

All fish sold after 11 years (Dec. 2008).

Total income $= 11 \times 15\,400 \times \$10$
$= \$1\,694\,000.$

(iv) Let I = total income
= sum of previous year's sales + sale of this year's entire stock
$= (n-1) \times 15\,400 \times 10 + F_n \times 15$
$= 154\,000(n-1) + 15\left[154\,000 - 54\,000(1{\cdot}1)^n\right]$
$= 154\,000n - 154\,000 + 2\,310\,000 - 810\,000(1{\cdot}1)^n$
$= 154\,000n - 810\,000(1{\cdot}1)^n + 2\,156\,000.$

We need to find maximum I for $n = 1$ to 11. Can be done by trial and error and/or by differentiation.

$$\frac{dI}{dn} = 0$$
$$154\,000 - 810\,000(1{\cdot}1)^n \ln(1{\cdot}1) = 0$$
$$\left(\frac{d}{dx}a^x = a^x \ln a\right)$$
$$\therefore \quad 810\,000(1{\cdot}1)^n \ln(1{\cdot}1) = 154\,000$$
$$(1{\cdot}1)^n = \frac{154}{810\ln(1{\cdot}1)}$$
$$\doteqdot 1{\cdot}9948.$$
$$\log(1{\cdot}1)^n = \log(1{\cdot}9948)$$
$$n\log(1{\cdot}1) = \log(1{\cdot}9948)$$
$$n = \frac{\log(1{\cdot}9948)}{\log(1{\cdot}1)}$$
$$\doteqdot 7{\cdot}2452 \text{ years.}$$

Test $n = 7$ or 8 years:

When $n = 7$, $I = \$1\,655\,539$
$n = 8$, $I = \$1\,651\,693.$

$\therefore$ Maximum total income occurs after 7 years. The farmer should sell after 7 years (December 2004).

END OF 2/3 UNIT (COMMON) MATHEMATICS SOLUTIONS