1997 HIGHER SCHOOL CERTIFICATE SOLUTIONS

2/3 UNIT (COMMON) MATHEMATICS

QUESTION 1

(a) $\dfrac{1}{7+5\times 3} = 0.0\dot{4}\dot{5}$

$= 0.0455$ (3 sig. figs)

(b) $(2-3x)-(5-4x) = 2-3x-5+4x$

$= x-3.$

(c) $180° = \pi^c$

$1° = \dfrac{\pi}{180}^c$

$\therefore\ 135° = 135\times\dfrac{\pi}{180}^c$

$= \dfrac{3\pi}{4}^c.$

(d) $PQ = r\theta$

$= 3\times\dfrac{5\pi}{6}$

$= \dfrac{5\pi}{2}$ cm.

(e) $\displaystyle\int \sec 3x\tan 3x\ dx = \frac{1}{3}\sec 3x + C.$

(f) There are 22 even number balls.

$\therefore$ P(even) $= \dfrac{22}{45}$

(g) $\dfrac{8}{3-\sqrt{5}}\times\dfrac{3+\sqrt{5}}{3+\sqrt{5}} = \dfrac{24+8\sqrt{5}}{9-5}$

$= 6+2\sqrt{5}.$

QUESTION 2

(a) (i) At L, $y = 0$

$\therefore\ 6x^2 - x^3 = 0$

$x^2(6-x) = 0$

$\therefore\ x = 0$ or $6.$

From the diagram the x coordinate at L is 6, $\therefore\ L = (6, 0).$

(ii) At M, $\dfrac{dy}{dx} = 0$ as M is a stationary point.

$\dfrac{dy}{dx} = 12x - 3x^2$

$= 3x(4-x)$

$= 0$ when $x = 0$ or $4.$

From the diagram the x coordinate at M is 4, $\therefore\ M = (4, 32).$

(iii) N is a point of inflection. $\therefore\ \dfrac{d^2y}{dx^2} = 0$ and there is a change in concavity.

$\dfrac{d^2y}{dx^2} = 12-6x = 0$ when $x = 2.$

Testing:

x	$2-\varepsilon$	2	$2+\varepsilon$
y''	+	0	−

$\therefore\ N = (2, 16).$

(b) $f'(x) = 2x+7$

$f(x) = x^2+7x+C$

$f(1) = 1+7+C = 4$

$C = -4$

$\therefore\ f(x) = x^2+7x-4.$

(c) $\displaystyle\int \frac{2x}{x^2+1}\,dx = \ln\left(x^2+1\right)+C.$

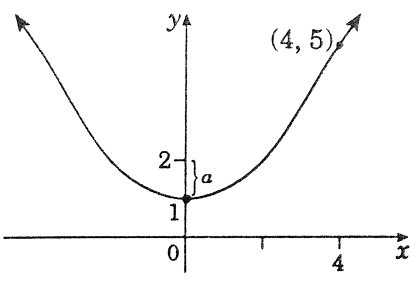
(d) We match $x^2 = 4(y-1)$ with the form

$(x-h)^2 = 4\times a\times(y-k),$

giving $(x-0)^2 = 4\times 1\times(y-1).$

(i) The vertex is at $(h, k) = (0, 1).$

(ii) Focal length = 1 unit $(\because\ a = 1).$



From the diagram, the focus is at (0, 2).

QUESTION 3

(a) (i) $\dfrac{d}{dx}\left(x^2+5\right)^3 = 3\times\left(x^2+5\right)^2\times 2x$

$= 6x\left(x^2+5\right)^2.$

(ii) $\dfrac{d}{dx}\left(\dfrac{\cos x}{x}\right) = \dfrac{x\times(-\sin x)-\cos x\times 1}{x^2}$

$= \dfrac{-x\sin x-\cos x}{x^2}.$

(iii) $\dfrac{d}{dx}\left(x^2\ln x\right) = 2x\ln x + x^2\times\dfrac{1}{x}$

$= 2x\ln x + x$

$= x(2\ln x+1).$

(b) (i) Midpoint of $AB = \left(\dfrac{0+2}{2}, \dfrac{1+3}{2}\right) = (1,2).$

(ii) Slope of $AB = \dfrac{3-1}{2-0} = \dfrac{2}{2} = 1.$

(iii) Use $y-y_1 = m(x-x_1)$, with $(x_1, y_1) = (1,2)$, and $m = -1$.

$\therefore \quad y-2 = -1\times(x-1)$

$\therefore \quad y = -x+1+2$

i.e. $\quad y = -x+3.$

Alternative method:
Points on the perpendicular bisectors are equidistant from A and B. So if $P(x, y)$ is on the perpendicular bisector, then

$PA^2 = PB^2.$

i.e. $(x-0)^2+(y-1)^2 = (x-2)^2+(y-3)^2$

x^2+y^2-2y+1

$= x^2-4x+4+y^2-6y+9$

$4y = -4x+12$

$y = -x+3.$

(iv) Points equidistant from A and B lie on the perpendicular bisector, so the coordinates of P are the simultaneous solutions of

$y = -x+3 \qquad -①$

and $\quad y = 2x-9. \qquad -②$

Equating ① and ② to eliminate y:

$\therefore \quad 2x-9 = -x+3$

$\therefore \quad 3x = 12$

$\therefore \quad x = 4.$

In ①, $\quad y = -4+3$

$= -1$

$\therefore \quad P = (4,-1).$

QUESTION 4

(a)

x	1	1·5	2	2·5	3
$f(x)$	5	1	−2	3	7

$\displaystyle\int_1^3 f(x)\,dx \approx \frac{2-1}{6}[5+(-2)+4\times 1]$

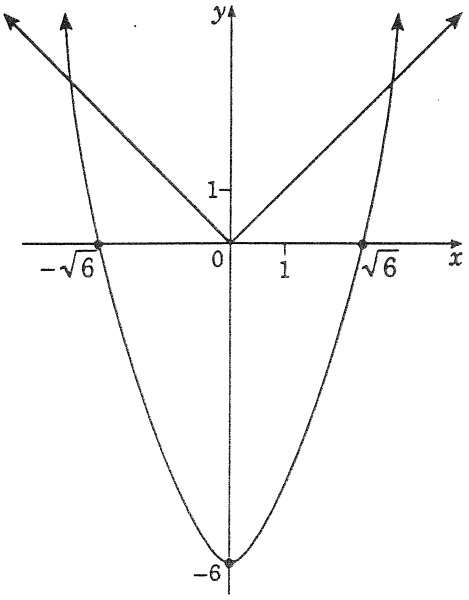
$+\dfrac{3-2}{6}(-2+7+4\times 3)$

$= \dfrac{1}{6}\times 7+\dfrac{1}{6}\times 17$

$= \dfrac{24}{6}$

$= 4.$

(b) (i) and (ii)



(iii) Solve $\quad x^2-6 = x \quad$ for $x>0$.

$x^2-x-6 = 0$

$(x-3)(x+2) = 0$

$\therefore \quad x = 3$ or $-2.$

Reject $x = -2$ (outside domain).

The x coordinates of the two points where the graphs intersect are 3 and −3.

(N.B. Both graphs are even functions.)

(iv) We want the x values for which the parabola is 'on' or below the straight lines, that is $-3 \le x \le 3$.

(c) $\text{Area} = \displaystyle\int_0^{\frac{\pi}{6}} \sin 2x\,dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos x\,dx$

$= \left[-\dfrac{\cos 2x}{2}\right]_0^{\frac{\pi}{6}} + \left[\sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= \dfrac{1}{2}\left(-\cos\dfrac{\pi}{3}+1\right)+\left(\sin\dfrac{\pi}{2}-\sin\dfrac{\pi}{6}\right)$

$= \dfrac{1}{2}\left(-\dfrac{1}{2}+1\right)+\left(1-\dfrac{1}{2}\right)$

$= \dfrac{1}{4}+\dfrac{1}{2}$

$= \dfrac{3}{4}.$

QUESTION 5

(a) $\displaystyle\int_0^{\ln 7} e^{-x}\,dx = \left[-e^{-x}\right]_0^{\ln 7}$

$= \left(-e^{-\ln 7} + e^0\right)$

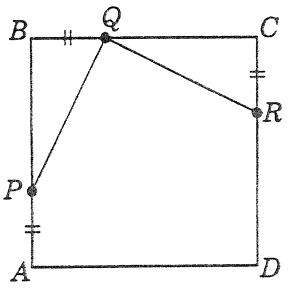
$= -e^{\ln 7^{-1}} + 1$

$= -7^{-1} + 1$

$= -\frac{1}{7} + 1$

$= \frac{6}{7}$.

(b)



(i) $BA = BC$ ($ABCD$ is a square)

$PA = BQ$ (given)

$\therefore\ BA - PA = BC - BQ$

i.e. $BP = QC$.

So in triangles PBQ and QCR, we have

$BP = QC$ (from above)

$BQ = CR$ (given)

$\angle PBQ = \angle QCR$ (both 90°, since $ABCD$ is a square)

$\Delta PBQ \equiv \Delta QCR$ (SAS).

(ii) $\angle BQP + \angle BPQ = 90°$ ($\angle PBQ = 90°$, angle sum of Δ)

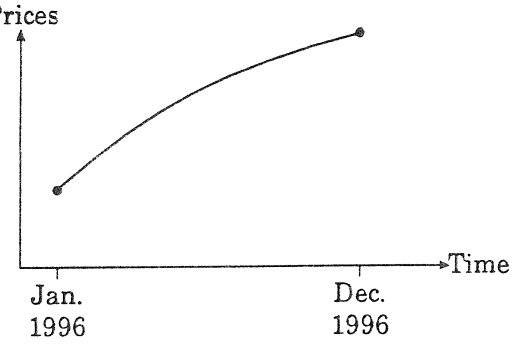
$\angle BPQ = \angle CQR$ (corresponding $\angle$s in congruent Δs)

$\therefore\ \angle BQP + \angle CQR = 90°$

$\therefore\ \angle PQR = 180° - 90°$ ($\angle CQR$, $\angle PQR$, $\angle BQP$ form a straight line.)

$= 90°$.

(c)



N.B. The curve is increasing and concave down (gradient is decreasing).

(d) (i) Height = 75% of 1·125 metres
= 0·843 75 metres.

(ii) This is a geometric sequence, with common ratio 0·75.

(iii) The total distance travelled is 2 m for the first drop, plus twice the new height for every bounce (since the ball goes up and down), that is $2 + 2(1{\cdot}5 + 1{\cdot}125 + \cdots)$.

If we leave out the first drop, we can add up all the heights (limiting sum in brackets above) using

$$S_\infty = \frac{a}{1-r},$$

with $a = 1{\cdot}5$ and $r = 0{\cdot}75$.

$$S_\infty = \frac{1{\cdot}5}{1 - 0{\cdot}75}.$$

Total distance travelled $= 2 + 2 \times \left(\frac{1{\cdot}5}{1 - 0{\cdot}75}\right)$

$= 14$ m.

QUESTION 6

(a) (i) $PQ + x = 5$

$\therefore\ PQ = (5 - x)$ m.

(ii) $PR^2 = (5-x)^2 - x^2$

$= 25 - 10x + x^2 - x^2$

$= 25 - 10x$.

$\therefore\ PR = \sqrt{25 - 10x}$ m.

Area: $A = \frac{1}{2} \times b \times h$

$= \frac{1}{2} \times x \times \sqrt{25 - 10x}$

$= \frac{1}{2} x \sqrt{25 - 10x}$ m^2.

(iii) Differentiate to find maximum.

$A' = \frac{1}{2}\sqrt{25 - 10x} + \frac{1}{2}x \cdot \frac{1}{2\sqrt{25 - 10x}} \cdot (-10)$

$= \frac{1}{2}\sqrt{25 - 10x} - \frac{1}{2} \cdot \frac{5x}{\sqrt{25 - 10x}}$

$= \frac{25 - 10x - 5x}{2\sqrt{25 - 10x}}$

$= \frac{25 - 15x}{2\sqrt{25 - 10x}}$.

$A' = 0$ when $25 - 15x = 0$, $\therefore\ x = \frac{5}{3}$.

If $x < \frac{5}{3}$, then $A' > 0$, and if $x > \frac{5}{3}$, then $A' < 0$, so this is a local maximum.

Since A is continuous on its domain $0 < x < 2{\cdot}5$, and since this is the only stationary point, this is also an absolute maximum.

$A_{\max} = \frac{1}{2} \times \frac{5}{3} \times \sqrt{25 - \frac{50}{3}}$

$= \frac{5}{6} \times \sqrt{\frac{25}{3}}$

$= \frac{25}{6\sqrt{3}} = \frac{25\sqrt{3}}{18}$ m^2.

(b) (i) Area of nth stripe
$= $ area of $(n+1)^{\text{th}}$ square
$-$ area of n^{th} square
$= [2(n+1)+4]^2-(2n+4)^2$
$= (2n+4+2)^2-(2n+4)^2$
$= (2n+4+2+2n+4)\times(2n+6-2n-4)$
$\left(\because\ a^2-b^2=(a+b)(a-b)\right)$
$= (4n+10)\times 2$
$= (8n+20)\text{ cm}^2.$

(ii) Area of 1st stripe $= (8\times 1+20)\text{ cm}^2$
$= 28\text{ cm}^2.$
Area of last stripe $(n=19) = 8\times 19+20$
$= 172\text{ cm}^2.$

(iii) There are 10 shaded stripes in arithmetic progression, so:
Area of shaded stripes
$=$ 1st + 3rd + 5th + $\cdots$ + 19th
$= 28+44+60+\cdots+172$
$= \frac{n}{2}(a+\ell)$
$= \frac{10}{2}(28+172)$
$= 5\times 200$
$= 1000\text{ cm}^2.$
Cost of gold paint $= \$9\times 1000$
$= \$9000.$

QUESTION 7

(a) $\sec\theta = \dfrac{1}{\cos\theta}$ and $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$\therefore\quad \sec^2\theta-\tan^2\theta = \dfrac{1}{\cos^2\theta}-\dfrac{\sin^2\theta}{\cos^2\theta}$
$= \dfrac{1-\sin^2\theta}{\cos^2\theta}$
$= \dfrac{\cos^2\theta}{\cos^2\theta}$
$= 1.$

OR

$\cos^2\theta+\sin^2\theta = 1$
$\therefore\quad 1-\sin^2\theta = \cos^2\theta$
$\therefore\quad \dfrac{1-\sin^2\theta}{\cos^2\theta} = 1.$
$\therefore\quad \dfrac{1}{\cos^2\theta}-\dfrac{\sin^2\theta}{\cos^2\theta} = 1$
$\therefore\quad \sec^2\theta-\tan^2\theta = 1.$

(b) (i) $V = \displaystyle\int_0^{\frac{\pi}{6}} \pi y^2\,dx$
$= \pi\displaystyle\int_0^{\frac{\pi}{6}} \tan^2 2x\,dx$
$= \pi\displaystyle\int_0^{\frac{\pi}{6}} \left(\sec^2 2x-1\right)dx$
$\left(\tan^2\theta=\sec^2\theta-1\text{ from (a)}\right).$

(ii) $V = \pi\left[\dfrac{1}{2}\tan 2x-x\right]_0^{\frac{\pi}{6}}$
$= \pi\left(\dfrac{1}{2}\tan\dfrac{\pi}{3}-\dfrac{\pi}{6}\right)-\pi(0-0)$
$= \pi\left(\dfrac{\sqrt{3}}{2}-\dfrac{\pi}{6}\right)\text{ units}^3.$

(c) (i) $v = Ce^{-kt}$
$\dfrac{dv}{dt} = C\times(-k)e^{-kt}$
$= -k\times Ce^{-kt}$
$= -kv.$

(ii) Using $v=100$ when $t=0$,
$100 = Ce^0$
$\therefore\quad C = 100.$

(iii) $\therefore\ v = 100e^{-kt}$
Using $v=85$ when $t=0{\cdot}25$
$85 = 100e^{-0\cdot 25k}$
$0{\cdot}85 = e^{-0\cdot 25k}$
$\therefore\quad -0{\cdot}25k = \ln(0{\cdot}85)$
$\therefore\quad k = -4\ln(0{\cdot}85)\quad (\approx 0{\cdot}650).$

(iv) $v = 100e^{-0\cdot 65t}$
$= 100e^{-1\cdot 3}$ when $t=2$
$= 27{\cdot}25\text{ m/s}.$

OR

$v = 100e^{-(-4\ln 0\cdot 85)t}$
$= 100e^{8\ln 0\cdot 85}$ when $t=2$
$= 100e^{\ln 0\cdot 85^8}$
$= 100\times 0{\cdot}85^8$
$= 27{\cdot}25\text{ m/s}.$

QUESTION 8

(a) (i) $\angle ADC$ is the exterior angle of ΔBAD.
$\therefore\ \angle ADC = \angle DBA+\angle DAB$ (exterior $\angle$ equals sum of int. opp. $\angle$s.)
$= \alpha+\beta.$

OR

$\angle BDA = 180° - (\alpha + \beta)$ ($\angle$ sum of ΔBDA).

$\therefore\ \angle ADC = 180° - \angle BDA$ (BDC is a straight line.)

$= 180° - [180 - (\alpha + \beta)]$

$= \alpha + \beta.$

(ii) $\angle DAC = 180° - (\alpha + \beta) - \alpha$ ($\angle$ sum of ΔADC)

$= 180° - 2\alpha - \beta$

$= 180° - (2\alpha + \beta).$

(iii) $\angle ADE = \dfrac{180° - \angle DAE}{2}$ ($\angle$ sum of ΔADE, base $\angle$s of isosceles Δ are equal.)

$= \dfrac{2\alpha + \beta}{2}$

$= \alpha + \dfrac{\beta}{2}.$

$\angle EDC = \angle ADC - \angle ADE$

$= \alpha + \beta - \left(\alpha + \dfrac{\beta}{2}\right)$

$= \dfrac{\beta}{2}.$

(b) (i) $x = t + \sin t, \quad 0 < t < 3\pi$

$\dfrac{dx}{dt} = 1 + \cos t.$

The particle is stationary when $\dfrac{dx}{dt} = 0.$

$\therefore\ \cos t = -1$

$\therefore\ t = \pi.$

(N.B. Also at $t = 3\pi$, but this is not in the domain.)

(ii) $\dfrac{d^2x}{dt^2} = -\sin t.$

Acceleration zero means $\dfrac{d^2x}{dt^2} = 0.$

$\therefore\ \sin t = 0$

$\therefore\ t = \pi$ or $2\pi.$

(N.B. Also at 0 and 3π, but again not in the domain.)

(iii) At $t = \pi$ we have a stationary point and a point of inflection.

$x = \pi + \sin \pi$

$= \pi \qquad (\pi, \pi).$

At $t = 2\pi$ we have a point of inflection.

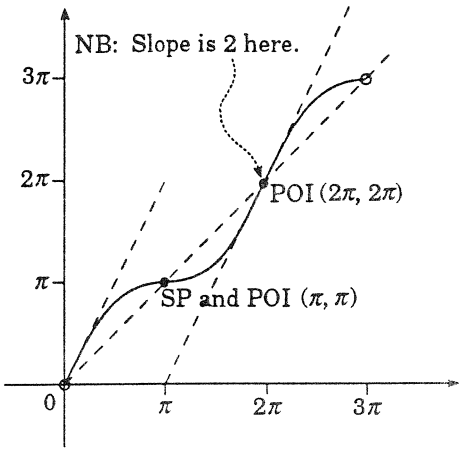
$x = 2\pi + \sin 2\pi$

$= 2\pi \qquad (2\pi, 2\pi).$

N.B. This is not a stationary point. It has a slope of $\dfrac{dx}{dt} = 1 + \cos 2\pi$

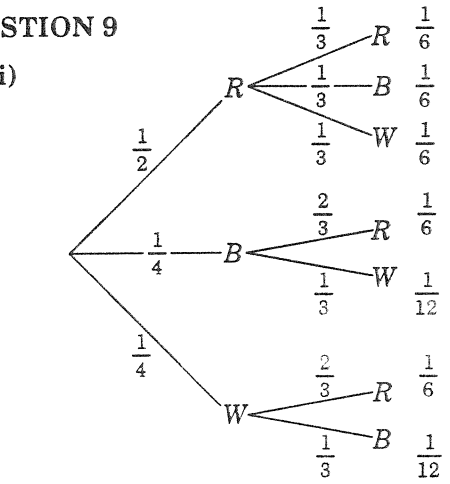
$= 1 + 1$

$= 2.$

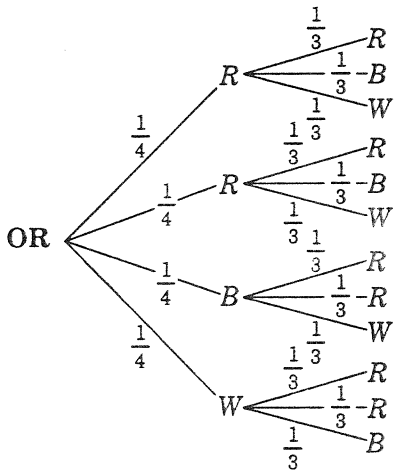


QUESTION 9

(a) (i)



OR



(ii) $P(\text{both red}) = \dfrac{1}{2} \times \dfrac{1}{3} = \dfrac{1}{6}.$

OR

$P(\text{both red}) = \dfrac{1}{12} + \dfrac{1}{12} = \dfrac{1}{6}.$

(iii) $P(\text{at least 1 red}) = \dfrac{10}{12} = \dfrac{5}{6}.$

OR

$P(\text{no red}) = P(BW) + P(WB)$

$= \dfrac{1}{4} \times \dfrac{1}{3} + \dfrac{1}{4} \times \dfrac{1}{3}$

$= \dfrac{1}{12} + \dfrac{1}{12} = \dfrac{1}{6}.$

$\therefore\ P(\text{at least 1 red}) = 1 - \dfrac{1}{6} = \dfrac{5}{6}.$

(iv) $P(2R \text{ given } 1R) = \frac{2}{10} = \frac{1}{5}$.

(b) (i) $\frac{dx}{dt} = 4t^3 - 24t^2 + 20t$.

$\therefore \quad x = t^4 - 8t^3 + 10t^2 + C$.

At $t = 0$, $x = 3$, $\therefore \quad C = 3$.

$x = t^4 - 8t^3 + 10t^2 + 3$.

(ii) At $t = 1$, $v = 0$ and changes sign from positive to negative:



So at $t = 1$, the $x = f(t)$ graph has a maximum stationary point.

(iii) At $t = 1$, $x = 1 - 8 + 10 + 3$
$= 6$.

At $t = 2$, $x = 16 - 64 + 40 + 3$
$= -5$.

Distance travelled (from $t = 0$ to 1)
$= 6 - 3 = 3$.

Distance travelled (from $t = 1$ to 2)
$= |-5 - 6| = 11$.

$\therefore$ Total dist. travelled = 14 cm.

Ink used = $0{\cdot}05 \times 14 = 0{\cdot}7$ milligrams.

OR

Distance travelled

$$= \int_0^1 v\,dt + \int_1^2 |v|\,dt$$

$$= \left[t^4 - 8t^3 + 10t^2\right]_0^1 + \left|\left[t^4 - 8t^3 + 10t^2\right]_1^2\right|$$

$$= (3 - 0) + |(-8 - 3)|$$

$$= 3 + 11$$

$$= 14 \text{ cm}.$$

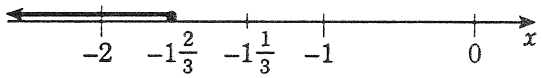
QUESTION 10

(a) $4x \le 15 \le -9x$.

i.e. $4x \le 15$ and $-9x \ge 15$.

$x \le \frac{15}{4} = 3\frac{3}{4}$ and $x \le -\frac{15}{9} = -1\frac{2}{3}$.

Since both conditions must be true, this simplifies to $x \le -1\frac{2}{3}$.



(b) (i) $OP = OQ$ (both radii equal to 1)

$\therefore$ OPQ is isosceles

$\therefore$ $\angle OPQ = \beta$ (base $\angle$s of isosceles Δ are equal)

$\therefore$ $\alpha = \angle OQP + \angle OPQ$ (exterior $\angle$ equals sum of interior opposite $\angle$s)
$= \beta + \beta$
$= 2\beta$.

(ii) PQ has slope $m = \tan\beta$ and goes through $Q(-1, 0)$.

$\therefore \quad y - 0 = m[x - (-1)]$

$y = m(x + 1)$

$y = mx + m$.

(iii) x coordinates of P and Q are the simultaneous solutions of

$$\left.\begin{aligned} y &= mx + m \\ x^2 + y^2 &= 1 \end{aligned}\right\}.$$

$\therefore \quad x^2 + [m(x+1)]^2 = 1$ (by substitution)

$\therefore \quad x^2 + m^2(x^2 + 2x + 1) = 1$

$\therefore \quad x^2 + m^2 x^2 + 2m^2 x + m^2 - 1 = 0$

$\therefore \quad (1 + m^2)x^2 + 2m^2 x + m^2 - 1 = 0$.

(iv) Equation in (iii) is a quadratic equation. x coordinates of $P(x_P)$ and $Q(-1)$ must multiply to give $\frac{c}{a}$, where $c = m^2 - 1$ and $a = 1 + m^2$.

$\therefore \quad -1 \times x_P = \frac{m^2 - 1}{1 + m^2}$

$\therefore \quad x_P = \frac{1 - m^2}{1 + m^2}$

$y_P = mx_P + m$

$$= m\left(\frac{1 - m^2}{1 + m^2} + 1\right)$$

$$= m\left(\frac{1 - m^2 + 1 + m^2}{1 + m^2}\right)$$

$$= \frac{2m}{1 + m^2}.$$

$\therefore \quad P = \left(\frac{1 - m^2}{1 + m^2}, \frac{2m}{1 + m^2}\right)$.

OR

$$x = \frac{-2m^2 \pm \sqrt{4m^4 - 4(m^2+1)(m^2-1)}}{2(m^2+1)}$$

$$= \frac{-2m^2 \pm \sqrt{4m^4 - (4m^4 - 4)}}{2(m^2+1)}$$

$$= \frac{-2m^2 \pm 2}{2(m^2+1)}$$

$$= \frac{-m^2 \pm 1}{m^2+1}$$

$$= \frac{-(m^2-1)}{m^2+1} \text{ or } \frac{-(m^2+1)}{m^2+1}$$

$$= \frac{-(m^2-1)}{m^2+1} \text{ or } -1.$$

Reject -1 as it is the x coordinate of Q.

$$\therefore \quad P = \left[\frac{1-m^2}{1+m^2},\ m\left(\frac{1-m^2}{1+m^2}\right)+m\right]$$

$$= \left[\frac{1-m^2}{1+m^2},\ \frac{m-m^3+m+m^3}{1+m^2}\right]$$

$$= \left[\frac{1-m^2}{1+m^2},\ \frac{2m}{1+m^2}\right].$$

(v) $\tan 2\beta = \tan\alpha$ (from (i))

$$= \frac{y_P}{x_P}$$

$$= \frac{2m}{1+m^2} \div \frac{1-m^2}{1+m^2}$$

$$= \frac{2m}{1-m^2}$$

$$= \frac{2\tan\beta}{1-\tan^2\beta}, \text{ since } m = \tan\beta.$$

END OF 2/3 UNIT (COMMON) MATHEMATICS SOLUTIONS