1996 HIGHER SCHOOL CERTIFICATE SOLUTIONS

2/3 UNIT (COMMON) MATHEMATICS

QUESTION 1

(a) $13^{-1.2} = 0{\cdot}046054\ldots$
$= 0{\cdot}046$ (correct to 2 sig. figs.)
N.B. Give calculator readout, then round.

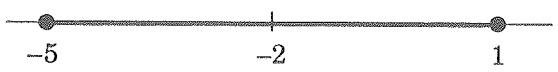
(b) $2x^2 + 3x - 2 = (2x-1)(x+2)$

(c) $\dfrac{4}{\sqrt{5}+2} \times \dfrac{\sqrt{5}-2}{\sqrt{5}-2} = \dfrac{4\left(\sqrt{5}-2\right)}{5-4}$
$= 4\left(\sqrt{5}-2\right)$

(d) $\dfrac{2}{3} - \dfrac{x-1}{4} = \dfrac{8-3(x-1)}{12}$
$= \dfrac{8-3x+3}{12}$
$= \dfrac{11-3x}{12}$

(e) Primitive of $6 - x^{-3} = 6x - \dfrac{x^{-2}}{-2} + c$
$= 6x + \dfrac{1}{2x^2} + c$

(f) $|x+2| \le 3$. That is, $|x-(-2)| \le 3$.
[Plot –2, and shade 3 units either side.]



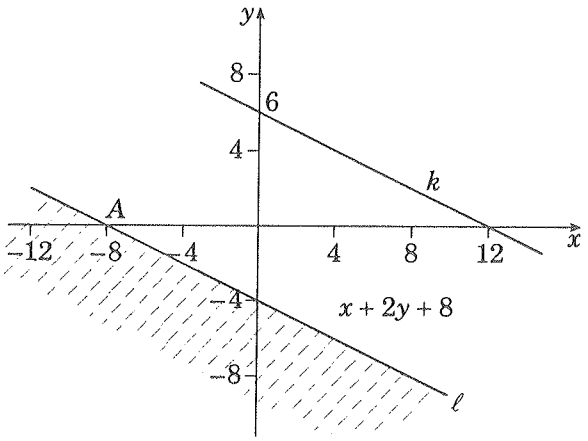
$-5 \le x \le 1$

OR

$-3 \le x+2 \le 3$. That is, $-5 \le x \le 1$.



QUESTION 2



(a) At A, $y = 0$
$\therefore$ Coordinates of A $(-8, 0)$.

(b) k is $y = -\dfrac{1}{2}x + 6$
$\therefore$ Gradient is $-\dfrac{1}{2}$ $(y = mx + b)$
ℓ is $x + 2y + 8 = 0$
$\therefore$ $2y = -x - 8$
$y = -\dfrac{1}{2}x - 4$
$\therefore$ Gradient is $-\dfrac{1}{2}$.
Since the gradients are the same, k is parallel to ℓ.

(c) See diagram.

(d) $x + 2y + 8 \le 0$. Test $(0, 0)$: $0 + 0 + 8 \le 0$
$\therefore$ False.

(e) Test $(0, 0)$ in $y + \dfrac{1}{2}x - 6 \le 0$
$0 + 0 - 6 \le 0$. $\therefore$ True.
$\therefore$ Inequalities are $\dfrac{1}{2}x + y - 6 \le 0$
and $x + 2y + 8 \ge 0$.
[*Note* The region between is $\frac{1}{2}x + y - 6 < 0$ and $x + 2y + 8 > 0$, but students were not penalised for including the lines (that is, the equals signs).]

(f) k is $y = -\frac{1}{2}x + 6$

If $x = 8$, $y = -\frac{1}{2} \times 8 + 6$

$= -4 + 6$

$= 2$

$\therefore$ $P(8, 2)$ lies on k.

(g) Perpendicular distance of P to ℓ

$$= \left| \frac{8 + 2 \times 2 + 8}{\sqrt{1^2 + 2^2}} \right| \text{ using } d = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$$

$$= \frac{20}{\sqrt{5}} \text{ units} = 4\sqrt{5} \text{ units.}$$

(h) Distance $PQ = \sqrt{(8-4)^2 + (2+6)^2}$

$= \sqrt{16 + 64}$

$= \sqrt{80}$

$= 4\sqrt{5}$ units.

Since the distance PQ is the same as the perpendicular distance, and the perpendicular distance is the shortest distance of a point from a line, then Q on ℓ is the point closest to P.

OR The gradient of $PQ = \frac{2 - (-6)}{8 - 4}$

$= \frac{8}{4}$

$= 2$

The gradients of both k and ℓ are $-\frac{1}{2}$, from (b), so PQ is perpendicular to both k and ℓ.

Hence, since the perpendicular distance is the shortest distance, Q is the point, on ℓ, and closest to P, on k.

QUESTION 3

(a) (i) $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right)$

$= \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{x}}$

(ii) $\frac{d}{dx}\left(xe^{2x}\right) = 1.e^{2x} + x.2e^{2x}$

$= e^{2x} + 2xe^{2x}$

(iii) $\frac{d}{dx}\left(\cos^2 x\right) = 2\cos x(-\sin x)$

$= -2\sin x \cos x$

OR $y = \cos^2 x$

$= u^2$ where $u = \cos x$

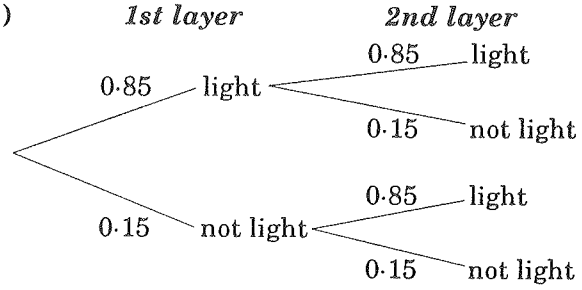
$\therefore$ $\frac{du}{dx} = -\sin x$ and $\frac{dy}{du} = 2u$

$\therefore$ $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= 2u \times -\sin x$

$= -2\sin x . \cos x.$

(b)



(i) Amount of light for 2 layers

$= 0{\cdot}85 \times 0{\cdot}85$

$= 0{\cdot}7225$

$= 72{\cdot}25\%$

(ii) If only 10% of light is allowed to pass, then there will be n sheets of plastic.

$0{\cdot}85^n \leq 0{\cdot}1$

$\therefore$ $n \log 0{\cdot}85 \leq \log 0{\cdot}1$

$n \geq \frac{\log 0{\cdot}1}{\log 0{\cdot}85}$

$\geq 14{\cdot}168$

$\therefore$ 15 layers are required.

(c) $\int \sec^2 6x \, dx = \frac{1}{6}\tan 6x + c$

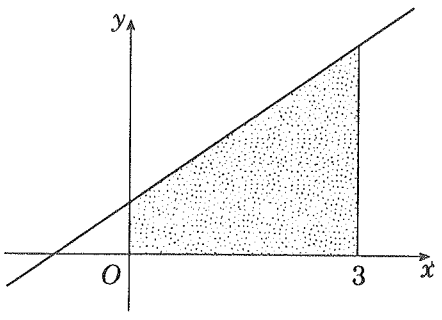
(d) $\int_1^{e^3} \frac{5}{x} \, dx = \left[5 \ln x\right]_1^{e^3}$

$= 5 \ln e^3 - 5 \ln 1$

$= 15$

QUESTION 4

(a)



$$\text{Volume} = \int_0^3 \pi y^2\,dx, \quad \text{where } y = x+1$$

$$= \pi\int_0^3 (x+1)^2\,dx$$

$$= \pi\left[\frac{(x+1)^3}{3}\right]_0^3$$

$$= \pi.\frac{4^3-1^3}{3}$$

$$= 21\pi.$$

OR $V = \pi\int_0^3 (x+1)^2\,dx$ using $V = \pi\int y^2\,dx$

$$= \pi\int_0^3 (x^2+2x+1)\,dx$$

$$= \pi\left[\frac{x^3}{3}+x^2+x\right]_0^3$$

$$= \pi\left[\frac{3^3}{3}+3^2+3\right]-\pi\left[\frac{0^3}{3}+0^2+0\right]$$

$$= 21\pi \text{ unit}^3.$$

OR $V = \frac{1}{3}\pi\left(R^3-r^3\right)$

$$= \tfrac{1}{3}\pi(64-1)$$

$$= \tfrac{1}{3}\times\pi$$

$$= 21\pi.$$

(b) $f(x) = \sqrt{25-x^2}$

(i)

x	0	1	2	3	4	5
$f(x)$	5·000	4·899	4·583	4·000	3·000	0·000
	y_0	y_1	y_2	y_3	y_4	y_5

(ii) $h = \dfrac{5-0}{5}$ using $h = \dfrac{b-a}{n}$

$\therefore\ h = 1$

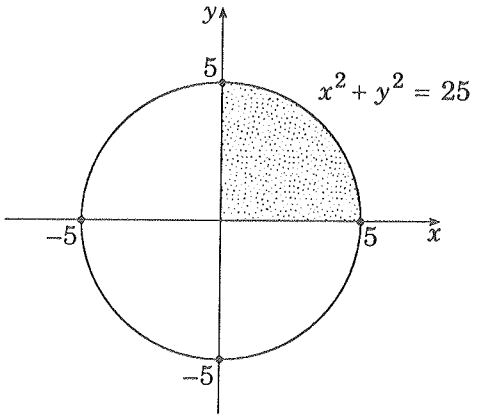
$$\int_0^5 \sqrt{25-x^2}\,dx$$

$$\approx \frac{h}{2}\left[y_0+y_5+2(y_1+y_2+y_3+y_4)\right]$$

$$\approx \frac{1}{2}\left[5+0+2(4{\cdot}899+4{\cdot}583+4+3)\right]$$

$$\approx 18{\cdot}982.$$

(iii)



(iv) This is a circle centre $(0, 0)$, radius 5 units.

Area of circle $= \pi\times 5^2$ using $A = \pi r^2$

$= 25\pi$ units^2.

But we have a quarter circle,

$\therefore$ area $= \dfrac{25\pi}{4}$ units^2.

(v) Using part (ii) and part (iv),

area $\approx$ 18·982 units^2.

But area $= \dfrac{25\pi}{4}$ units^2

$\therefore\ \dfrac{25\pi}{4} \approx 18{\cdot}982$

$\therefore\ \pi \approx \dfrac{18{\cdot}982\times 4}{25}$

$\approx 3{\cdot}037.$

QUESTION 5

(a) $f'(x) = 3x^2-2, \quad \therefore\ f(x) = \dfrac{3x^3}{3}-2x+c$

Since $(1, 3)$ satisfies this function,

$3 = 1-2+c$

$\therefore\ c = 4$

$\therefore\ f(x) = x^3-2x+4.$

(b) $u^2-u-1 = 0$

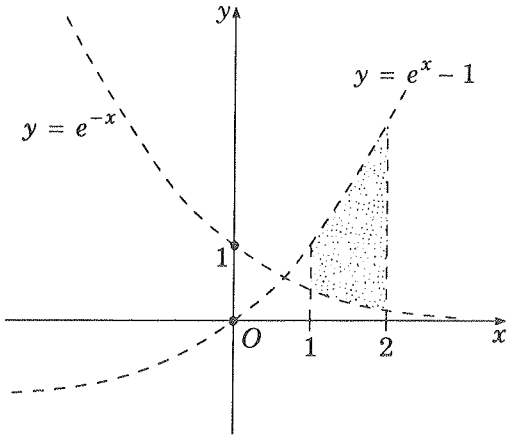
$$u = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$= \frac{1\pm\sqrt{1+4}}{2}$$

$$= \frac{1\pm\sqrt{5}}{2}$$

$= 1{\cdot}618$ or $-0{\cdot}618$, correct to 3 dec. places.

(c)



(i) Area $= \int_1^2 \left(e^x - 1 - e^{-x} \right) dx$

$= \left[e^x - x + e^{-x} \right]_1^2$

$= e^2 - 2 + e^{-2} - \left(e - 1 + e^{-1} \right)$

$= e^2 + e^{-2} - e - e^{-1} - 1$ units^2.

(ii) Solving $y = e^x - 1$ and $y = e^{-x}$

$\therefore \quad e^x - 1 = e^{-x}$,

that is, $e^x - 1 = \dfrac{1}{e^x}$

$e^{2x} - e^x - 1 = 0$.

(iii) $u^2 - u - 1 = 0$ has solution

$u \approx 1{\cdot}618$ or $u \approx -0{\cdot}618$

$\therefore \quad e^x \approx 1{\cdot}618$ or $e^x \approx -0{\cdot}618$.

But $e^x > 0$

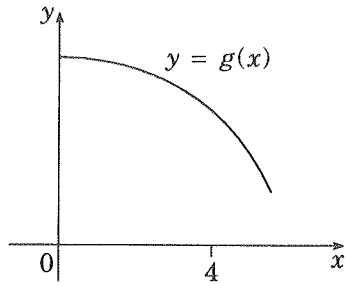
$\therefore \quad e^x \approx 1{\cdot}618$

$\therefore \quad x \approx \ln 1{\cdot}618$

that is, $x \approx 0{\cdot}481$.

(d) Since $g'(x) < 0$, curve is decreasing, that is, has a negative gradient.

Since $g''(x) < 0$, curve is concave down, that is:



The above graph is a possible solution.

QUESTION 6

(a) $M = 4t^2 - t^3, \qquad 0 \le t \le 3$

$= t^2 (4 - t)$

$\dfrac{dM}{dt} = 8t - 3t^2 = 0$ at stationary points

$\therefore \quad t(8 - 3t) = 0$

$t = 0, \dfrac{8}{3}$.

When $t = \dfrac{8}{3}$, $M = 4 . \dfrac{64}{9} - \dfrac{512}{27}$

$= \dfrac{768}{27} - \dfrac{512}{27}$

$= \dfrac{256}{27}$

$= 9{\cdot}48\dot{1}$

$t = 0$, $M = 0$, $\dfrac{d^2M}{dt^2} = 8 - 6t$.

When $t = \dfrac{8}{3}$, $\dfrac{d^2M}{dt^2} = 8 - 16$

$= -8$

< 0,

$\therefore$ maximum at $\left(\dfrac{8}{3}, 9{\cdot}48\dot{1} \right)$.

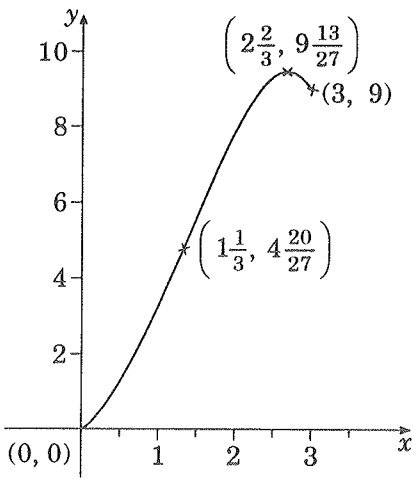
When $t = 0$, $\dfrac{d^2M}{dt^2} = 8$

> 0,

$\therefore$ minimum at $(0, 0)$.

End points are $(0, 0)$ and $(3, 9)$.

(i)



(ii) Medicine a maximum at $2\frac{2}{3}$ hours, or 2 hours 40 minutes.

(iii) The maximum value of $\dfrac{dM}{dt}$ is when

$\dfrac{d^2M}{dt^2} = 0$

$8 - 6t = 0$,

that is, $t = \dfrac{8}{6} = \dfrac{4}{3}$,

that is, after $1\frac{1}{3}$ hours.

(b) **(i)** Total thickness of closed blinds
$= 25 \times 3$ mm
$= 75$ mm.
Total height of opened blind
$= 25 \times (3 + 27$ mm$)$
$= 25 \times 30$ mm
$= 750$ mm.
$\therefore$ Bottom slat rises $(750 - 75)$ mm, that is, 675 mm.

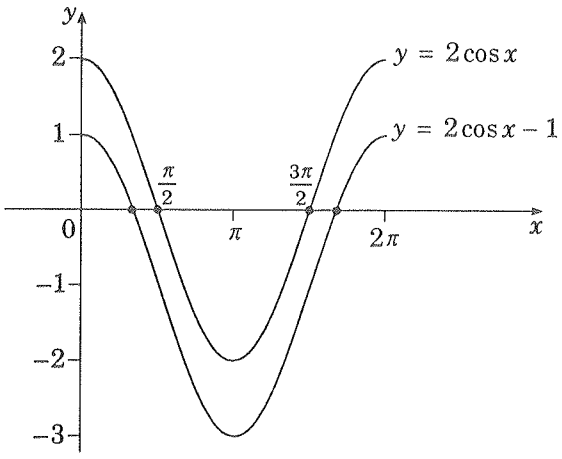
(ii) Bottom of 24th slat is 30×24 mm down when opened, that is, 24th slat is 720 mm down when opened.
When closed, 24th slat is 24×3 mm down, that is, 72 mm down.
$\therefore$ 2nd bottom slat rises $(720 - 72)$ mm, that is, 648 mm.

(iii) The bottom of the nth slat down is
(a) $n \times 30$ mm down when open,
(b) $n \times 3$ mm down when closed.
$\therefore$ The nth slat rises $30n - 3n$, that is, $27n$.
$\therefore$ 1st slat rises $27 \times 1 = 27$ mm
2nd " " $27 \times 2 = 54$ mm
3rd " " $27 \times 3 = 81$ mm,
etc., which is an AP; $a = 27$, $d = 27$.

(iv) Total distance
$$= \frac{25}{2}\left[2 \times 27 + (25 - 1) \times 27\right]$$
using $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$
$= 8775$ mm.

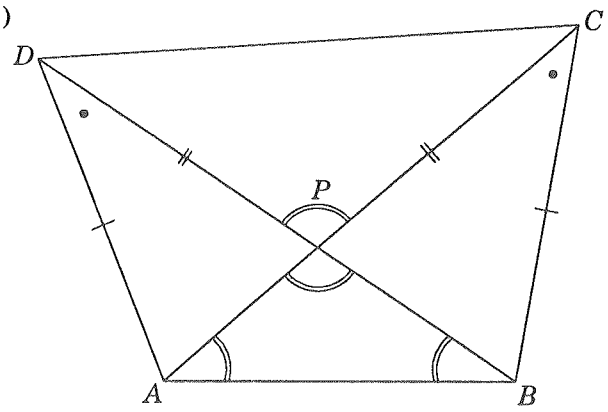
QUESTION 7

(a) **(i)** and **(ii)**



(iii) $2\cos x - 1 = 0$
$\therefore \quad \cos x = \frac{1}{2}$
$\therefore \quad x = \frac{\pi}{3}, \frac{5\pi}{3}$.

(b)



(i) In $\triangle ABC$, $\triangle ABD$,
$BC = AD$ (given)
$AC = BD$ (given)
AB is common.
$\therefore$ The triangles are congruent (SSS).

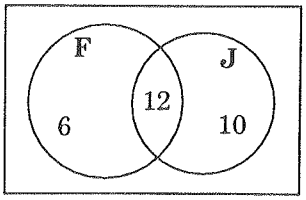
(ii) $\therefore \angle CAB = \angle DBA$ (corres. angles in congruent triangles)
$\therefore \triangle APB$ is isosceles, two equal angles.

(iii) $\therefore \quad AP = PB$
But $AC = BD$ (given)
$\therefore$ by subtraction, $PC = PD$
$\therefore \triangle PCD$ is isosceles.

(iv) Let $\angle DPC = x°$
$\therefore \quad \angle APB = x°$ (vertically opposite to $\angle DPC$).
In $\triangle APB$, $\angle PAB = \frac{1}{2}(180° - x°)$
(angle sum of isosceles triangle).
Similarly $\angle DCP = \frac{1}{2}(180° - x°)$
$\therefore \quad \angle PAB = \angle DCP$.
Since these are equal alternate angles,
$\therefore \quad DC \parallel AB$.

QUESTION 8

(a) Since there are only 28 students, and 18 French students plus 22 Japanese total 40, there are $40 - 28 = 12$ students who study both languages.



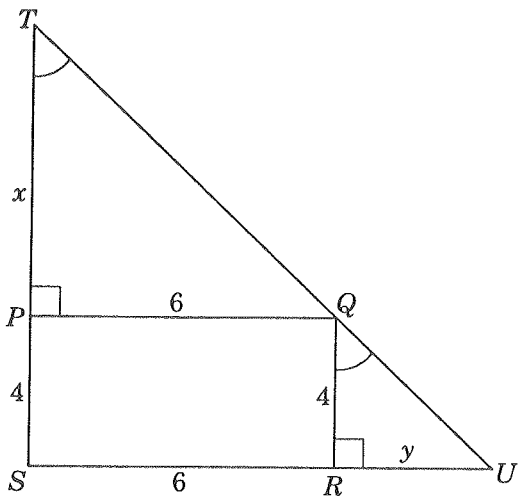
(i) Probability (studies French)
$$= \frac{18}{28} = \frac{9}{14}.$$

(ii) Probability (both study French)

$$= \frac{18}{28} \times \frac{17}{27} = \frac{17}{42}.$$

(iii) Probability of studying both languages $= \frac{12}{28} = \frac{3}{7}$.

(b)



(i) In ΔTPQ, ΔQRU,

$PS \parallel QR$ (opp. sides rectangle)

$\therefore$ $\angle STQ = \angle RQU$ (corres. $\angle$s, $PS \parallel QR$)

$\angle TPQ = \angle QRU = 90°$ (angles adjacent to 90° angles of the rectangle)

$\therefore$ The triangles are equiangular, hence similar.

(ii) $\therefore \quad \frac{TP}{QR} = \frac{PQ}{RU}$ (corres. sides are proportional)

$\therefore \quad \frac{x}{4} = \frac{6}{y}$

$\therefore \quad xy = 24.$

(iii) Area $A = \frac{1}{2} TS.SU$

$$= \frac{1}{2}(4+x)(6+y)$$

$$= \frac{1}{2}(24+4y+6x+xy)$$

$$\therefore \quad A = \frac{1}{2}\left(24 + 4.\frac{24}{x} + 6x + 24\right) \qquad \left(xy = 24, \ y = \tfrac{24}{x}\right)$$

$$= 24 + \frac{48}{x} + 3x$$

$$= 24 + 48x^{-1} + 3x.$$

(iv) $\frac{dA}{dx} = -48x^{-2} + 3$

$$= -\frac{48}{x^2} + 3$$

$= 0$ when $\frac{48}{x^2} = 3$

$$x^2 = 16$$

$$x = \pm 4$$

(but $x > 0$ since x is a length)

$\therefore \quad x = 4, \ y = 6$

$\frac{d^2A}{dx^2} = 96x^{-3} > 0$ for all $x > 0$,

$\therefore$ min. value for $x = 4$

$\therefore$ $TS = 8$ cm, $SU = 12$ cm.

QUESTION 9

(a) $v = 2t + 4, \quad \therefore \ x = \frac{2t^2}{2} + 4t + c$

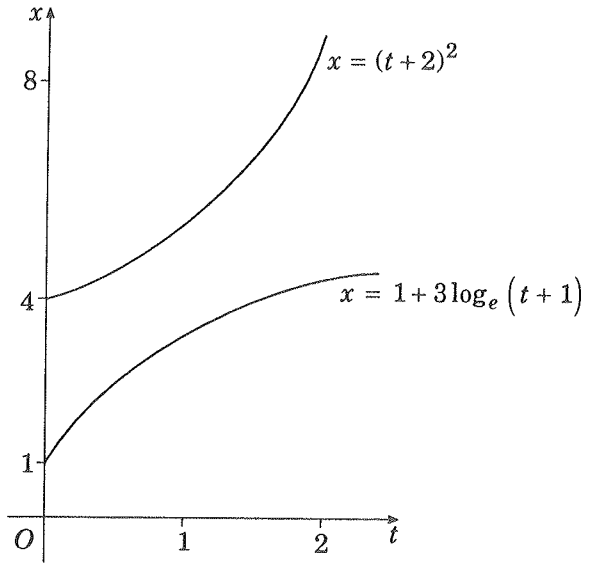
But when $t = 0$, $x = 4$

$\therefore \quad c = 4$

$\therefore \quad x = t^2 + 4t + 4$

$$= (t+2)^2.$$

(b)



(c) The x coordinate of M is given by the average of the two x values of the endpoints,

that is, $x = \frac{1}{2}\left[t^2 + 4t + 4 + 1 + 3\ln(t+1)\right]$

$$= \frac{1}{2}\left[t^2 + 4t + 3\ln(t+1) + 5\right].$$

(d) $\therefore \quad \frac{dx}{dt} = \frac{1}{2}\left[2t + 4 + \frac{3}{t+1}\right]$

$$= t + 2 + \frac{3}{2(t+1)}$$

$$= t + 2 + \frac{3}{2}(t+1)^{-1}$$

$$\frac{d^2x}{dt^2} = 1 - \frac{3}{2}(t+1)^{-2}$$

$$= 1 - \frac{3}{2(t+1)^2}$$

$= 0$ for acceleration zero.

That is, when $\dfrac{3}{2(t+1)^2} = 1$

$$3 = 2(t+1)^2$$
$$= 2\left(t^2+2t+1\right)$$
$$= 2t^2+4t+2$$
$$\therefore \quad 2t^2+4t-1 = 0$$
$$\therefore \quad t = \frac{-4\pm\sqrt{16+8}}{4}$$
$$= \frac{-4\pm 2\sqrt{6}}{4}$$
$$= -1\pm\sqrt{\frac{3}{2}}.$$

Since $t \geq 0$,

$t = -1+\sqrt{\dfrac{3}{2}}$ when acceleration is zero.

OR $3 = 2(t+1)^2$

$$\therefore \quad (t+1)^2 = \frac{3}{2}$$
$$t+1 = \pm\sqrt{\frac{3}{2}}$$
$$t = -1\pm\sqrt{\frac{3}{2}} \quad \text{etc.}$$

(e) Distance between P and Q is

$$D = t^2+4t+4-\left[1+3\ln(t+1)\right]$$
$$= t^2+4t-3-3\ln(t+1).$$

We note that initially, when $t=0$, distance is 3 units apart.

Checking for minimum, $\dfrac{dD}{dt} = 2t+4-\dfrac{3}{t+1}$

$$= 0.$$

When $(2t+4)(t+1) = 3$

that is, $2t^2+6t+1 = 0$

$$t = \frac{-6\pm\sqrt{36-8}}{4}$$
$$= \frac{-6\pm 2\sqrt{7}}{4}$$
$$= \frac{-3\pm\sqrt{7}}{2}.$$

Both values of t are negative and so are outside the domain. The two curves are moving apart as $t\to\infty$.

$\therefore$ Minimum distance is the initial value, 3 units.

OR For $x = t^2+4t+4$

$$\frac{dx}{dt} = 2t+4.$$

As t increases, the slope increases.

For $x = 1+3\log_e(t+1)$

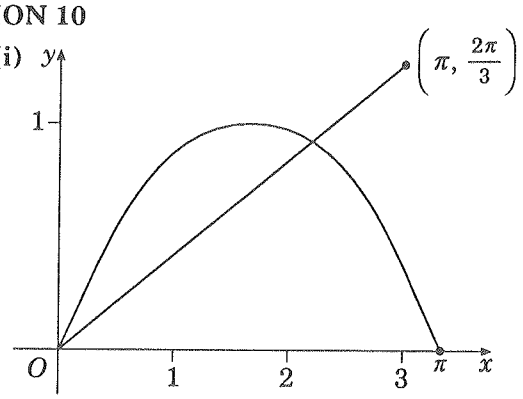
$$\frac{dx}{dt} = \frac{3}{t+1}.$$

As t decreases, the slope decreases.

$\therefore$ The curves diverge, and the minimum distance will be when $t=0$, that is, 3 units.

QUESTION 10

(a) (i)



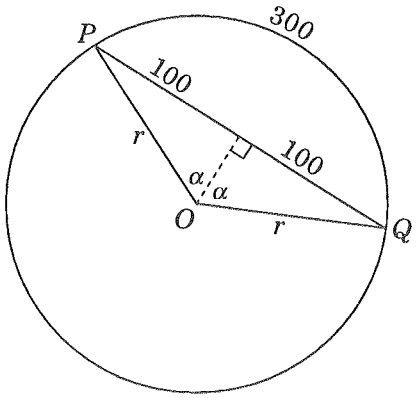
(ii) $y = \sin x, \quad \dfrac{dy}{dx} = \cos x.$

When $x=0$, $\cos 0 = 1$.

$\therefore$ Gradient at the origin is 1.

(iii) For a solution of $\sin x = mx$, we need $y = mx$ to intersect $y = \sin x$. This will occur when $0 < m < 1$ for the domain $0 < x < \pi$.

(b)



(i) Let the radius be r metres.

Then since arc $= r\theta$

$$\therefore \quad 300 = r\times 2\alpha$$

that is, $r = \dfrac{150}{\alpha}$.

Draw the line from O, perpendicular to PQ,

then $\sin\alpha = \dfrac{100}{r}$,

that is, $\sin\alpha = \dfrac{100}{150}\times\alpha$

$$= \frac{2\alpha}{3}.$$

(ii) From the graph in (a), $\alpha \approx 1{\cdot}5$.

(iii) $\therefore \angle POQ = 2\alpha = 3{\cdot}0$ radians

and $r = \dfrac{150}{1{\cdot}5} = 100.$

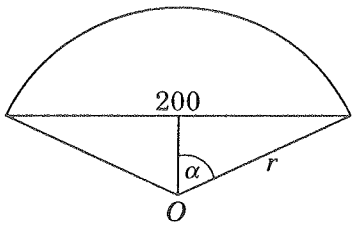
(c) Given AB is 200 m.

If there is no requirement that the arc be that of a minor segment, then ℓ can be as large as we please, that is, $\ell \to \infty$.

However, $\ell > 200$ for all values of the radius, since the arc must be greater than the straight line joining AB.

OR

Using the earlier parts of the question:



$$\sin\alpha = \frac{100}{r} \Rightarrow r = \frac{100}{\sin\alpha}.$$

But $\ell = r\theta$

$$\therefore \quad \ell = \frac{100}{\sin\alpha} \times 2\alpha \quad \text{since } \theta = 2\alpha$$

$$\therefore \sin\alpha = \frac{200}{\ell} \times \alpha.$$

But if $\sin x = mx$, then $0 < m < 1$

$$\therefore \quad 0 < \frac{200}{\ell} < 1$$

$$\therefore \quad 0 < 200 < \ell$$

$\therefore$ Solution is $\ell > 200$.

END OF 2/3 UNIT MATHEMATICS SOLUTIONS