1995 HIGHER SCHOOL CERTIFICATE SOLUTIONS

2/3 UNIT (COMMON) MATHEMATICS

QUESTION ONE

(a) $9x^2-16=(3x)^2-4^2$
$=(3x-4)(3x+4)$

(b) $0{\cdot}2294\ldots$
$\approx 0{\cdot}23$ (2 dec. places)

(c) π radians $=180°$
$\therefore \frac{3\pi}{5}$ radians $=\frac{3}{5}\times 180°$
$=108°.$

(d) $\frac{x}{3}+\frac{3x-1}{2}=\frac{2x}{6}+\frac{3(3x-1)}{6}$
$=\frac{2x+9x-3}{6}$
$=\frac{11x-3}{6}$

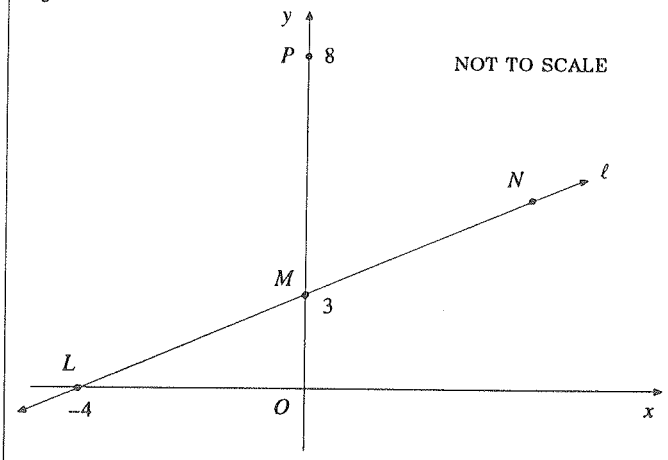
(e) $\int(2x+11)\,dx=x^2+11x+\text{constant}$

(f) Let $x=0{\cdot}2323\ldots$ —①
$\therefore\ 100x=23{\cdot}2323\ldots$ —②
②−①:
$99x=23$
$x=\frac{23}{99}$
i.e. $0{\cdot}\dot2\dot3=\frac{23}{99}$

OR $0{\cdot}\dot2\dot3=232\,323\ldots$
$=0{\cdot}23+0{\cdot}0023+0{\cdot}000\,023+\cdots$
$\therefore$ Infinite sum of a geometric progression,
where $a=0{\cdot}23,\ r=\frac{0{\cdot}0023}{0{\cdot}23}=0{\cdot}01$
$\therefore\ S=\frac{a}{1-r}=\frac{0{\cdot}23}{1-0{\cdot}01}$
$=\frac{0{\cdot}23}{0{\cdot}99}=\frac{23}{99}$
i.e. $0{\cdot}\dot2\dot3=\frac{23}{99}.$

(g) $5-3x<7$
$\therefore\ -3x<2$
$3x>-2$
$\therefore\ x>-\frac{2}{3}$

QUESTION TWO



(a) Gradient of line $\ell=\frac{3-0}{0-(-4)}=\frac{3}{4}.$

OR Slope $=\frac{\text{rise}}{\text{run}}=\frac{OM}{LO}=\frac{3}{4}$

Equation of line ℓ:
Using $y=mx+b$
$\therefore\ y=\frac{3}{4}x+3$

OR using $y-y_1=m(x-x_1)$
with $(x_1,y_1)=(-4,0)$
$\therefore\ y-0=\frac{3}{4}(x+4)$
$y=\frac{3}{4}x+3.$

(b) Test $(16,15)$ on $y=\frac{3}{4}x+3.$
LHS $=y=15$
RHS $=\frac{3}{4}x+3=\frac{3}{4}\times 16+3=15=$ LHS
$\therefore$ $(16,15)$ lies on the line ℓ.

(c) By inspection, length of $MP = 8-3$ units
$= 5$ units.

Using distance formula, or Pythagoras' Theorem, length of $ML = \sqrt{3^2+4^2}$ units
$= 5$ units

$\therefore \quad ML = MP.$

That is, ΔLMP is isosceles.

(d) $P(0,8)$ and $L(-4,0)$:

$\therefore \quad$ Gradient $= \dfrac{8-0}{0-(-4)}$
$= 2.$

(e) Let $N = (x, y)$

$\therefore$ Using the midpoint formula:

$$\frac{x+(-4)}{2} = 0$$

$$\therefore \quad x = 4$$

and $$\frac{y+0}{2} = 3$$

$$\therefore \quad y = 6$$

$\therefore N$ has coordinates $(4, 6)$.

(f) Gradient $PL = 2$, from (d),

gradient $PN = \dfrac{8-6}{0-4} = -\dfrac{1}{2}$

$\therefore$ gradient $PL \times$ gradient $PN = 2 \times -\dfrac{1}{2} = -1.$

$\therefore PL \perp PN.$

That is, $\angle NPL$ is a right angle.

OR

Using the converse of Pythagoras' theorem:

Length $LN = \sqrt{\left(4-(-4)\right)^2+(6-0)^2}$ units
$= 10$ units.

Length $LP = \sqrt{(-4)^2+(-8)^2}$ units
$= 4\sqrt{5}$ units.

Length $NP = \sqrt{(4-0)^2+(6-8)^2}$ units
$= 2\sqrt{5}$ units.

$\therefore \quad LN^2 = 100$

$LP^2 + NP^2 = 80 + 20$
$= 100$
$= LN^2.$

$\therefore \Delta NPL$ is right angled at P.

That is, $\angle NPL$ is a right angle.

(g) $ML = MN = MP = 5$ units

That is, ML, MN, and MP are all (equal) radii of circle with centre at $M(0,3)$ and radius 5 units.

$\therefore \quad (x-0)^2+(y-3)^2 = 5^2$

i.e. $\quad x^2+(y-3)^2 = 25.$

N.B. (f) could have been deduced from (g) using the property that an angle in a semi-circle is a right angle.

QUESTION THREE

(a) (i) $\dfrac{d}{dx}(2x^{-3}) = -6x^{-4}$

(ii) $\dfrac{d}{dx}(4e^{2x}) = 4 \times 2e^{2x}$
$= 8e^{2x}$

(iii) Let $u = x$ and $v = \log_e x$

$\therefore \dfrac{du}{dx} = 1 \qquad \dfrac{dv}{dx} = \dfrac{1}{x}$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$= \log_e x + x \times \frac{1}{x}$$

i.e. $\dfrac{d}{dx}(x\log_e x) = \log_e x + 1.$

(b) (i) In ΔQRS,

$$\sin 52° = \frac{8}{SR}$$

$$\therefore \quad SR = \frac{8}{\sin 52°}$$

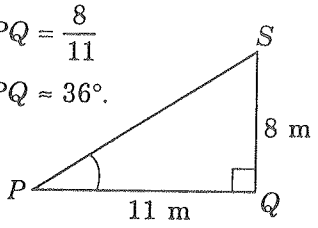
$\therefore \quad SR \approx 10{\cdot}15$ m (to 2 dec. places).

(ii) Since P is the midpoint of AQ, then $PQ = 11$ m.

In ΔPQS,

$$\tan \angle SPQ = \frac{8}{11}$$

$\therefore \quad \angle SPQ \approx 36°.$



(c) (i) $\displaystyle\int e^{3x}\,dx = \frac{1}{3}e^{3x} + c$

(ii) $\displaystyle\int_0^{\frac{\pi}{2}} \sin 2x\,dx = -\frac{1}{2}\Big[\cos 2x\Big]_0^{\frac{\pi}{2}}$

$$= -\frac{1}{2}(\cos\pi - \cos 0)$$
$$= -\frac{1}{2}(-1-1)$$
$$= 1.$$

QUESTION FOUR

(a) $a = 30,\ \ell = 50,\ S_n = 1240$

(i) Need to find n where

$$S_n = \frac{n}{2}(a+\ell)$$

$$\therefore \quad 1240 = \frac{n}{2}(30+50)$$

$$\therefore \quad 1240 = \frac{n}{2}\times 80$$

$$\therefore \quad 40n = 1240$$

$$n = 31.$$

$\therefore$ There are 31 strings.

(ii) Need to find d where

$$T_n = a+(n-1)d$$

Now, $T_{31} = 50$

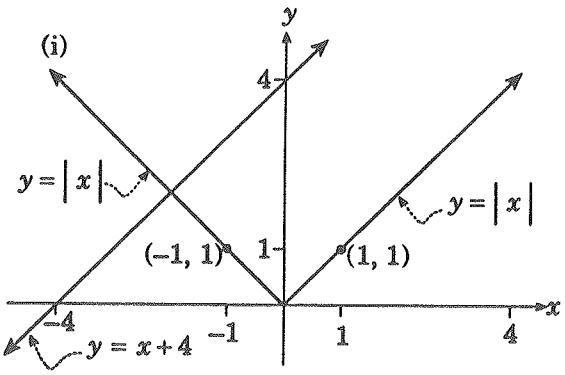
$$\therefore \quad 50 = 30+30\times d$$

$$\therefore \quad 30d = 20$$

$$\therefore \quad d = \frac{2}{3}.$$

$\therefore$ Difference in length is $\frac{2}{3}$ cm.

(b) (i)



(ii) *Method 1*:

$y = |x|$ is made up of two branches:

the right-hand branch is $y = x$ and the left-hand branch is $y = -x$.

$\therefore$ Need to consider solution of:

$$y = -x \qquad \text{—①}$$

$$y = x+4 \qquad \text{—②}$$

Subst. for $y = -x$ from ① into ②:

$$\therefore \quad -x = x+4$$

$$\therefore \quad -2x = 4$$

$$x = -2$$

$\therefore\ x = -2,\ y = 2.$

Method 2:

$\therefore$ Two graphs intersect at $(-2, 2)$.

The triangle formed by the x axis, $y = |x|$ and $y = x+4$ in the second quadrant is isosceles because $y = x+4$ and $y = |x|$ make angles of 45° with the x axis.

$\therefore$ The intersection of $y = x+4$ and $y = |x|$ occurs on the axis of symmetry of the triangle, $x = -2$.

$\therefore$ The point of intersection is $(-2, 2)$.

(c) (i)

x	-1	0	1	2	3
2^x	$\frac{1}{2}$	1	2	4	8

(ii) $$\text{Area} \approx \frac{1-(-1)}{6}\left(\frac{1}{2}+2+4\times 1\right) + \frac{3-1}{6}\left(2+8+4\times 4\right)$$

$$= \frac{1}{3}\times 6\frac{1}{2} + \frac{1}{3}\times 26$$

$$= 10\frac{5}{6} \text{ units}^2.$$

QUESTION FIVE

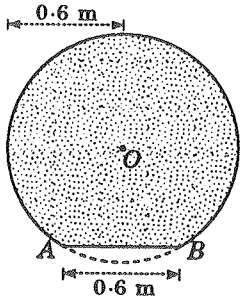
(a) (i) $AO = OB = 0{\cdot}6$ metres (equal radii)

$AB = 0{\cdot}6$ m (data)

$\therefore\ \Delta AOB$ is equilateral

$\therefore\ \angle AOB = \frac{\pi}{3}.$

(ii) *Method 1*:



Area of whole circle $= \pi\times 0{\cdot}6^2 \text{ m}^2$

$$= 0{\cdot}36\pi \text{ m}^2.$$

Area of sector $AOB = \frac{1}{2}r^2\theta$

$$= \frac{1}{2}\times 0{\cdot}6^2\times\frac{\pi}{3}$$

$$= \frac{3\pi}{50} \text{ m}^2.$$

Area of $\Delta AOB = \frac{1}{2}ab\sin O$

$$= \frac{1}{2}\times 0{\cdot}6^2\times\sin\frac{\pi}{3}$$

$$= \frac{9\sqrt{3}}{100} \text{ m}^2.$$

$\therefore$ Area of minor segment (i.e., part of circle removed)

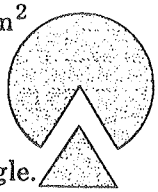
$$= \left(\frac{3\pi}{50} - \frac{9\sqrt{3}}{100}\right) \text{m}^2$$

$$\approx 0{\cdot}0326 \text{ m}^2.$$

$\therefore$ Area of table-top

$\approx (0{\cdot}36\pi - 0{\cdot}0326)\ \text{m}^2$

$\approx 1{\cdot}098\ \text{m}^2$.



Method 2:

Consider the area as a major sector plus a triangle.

Sector angle $= 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.

$\therefore$ Area

$$= \frac{1}{2}r^2\theta + \frac{1}{2}ab\sin c$$

$$= \frac{1}{2}\times 0{\cdot}6^2 \times \frac{5\pi}{3} + \frac{1}{2}\times 0{\cdot}6^2 \times \sin\frac{\pi}{3}$$

$\approx 1{\cdot}098\ \text{m}^2$.

(b) (i) $y = 7 + 4x^3 - 3x^4$

$\therefore \quad \frac{dy}{dx} = 12x^2 - 12x^3$

$\frac{dy}{dx} = 0$ for stationary points

$\therefore \quad 12x^2 - 12x^3 = 0$

$x^2(1-x) = 0$

$\therefore \quad x = 0$ or $x = 1$.

When $x = 0$,

$y = 7 + 4\times 0 - 3\times 0 = 7$

and when $x = 1$,

$y = 7 + 4 - 3 = 8$.

$\therefore$ Stationary points are $(0, 7)$ and $(1, 8)$.

(ii) $\frac{d^2y}{dx^2} = 24x - 36x^2$

$\therefore \quad 24x - 36x^2 = 0$

i.e. $\quad 2x - 3x^2 = 0$

$x(2-3x) = 0$

$\therefore \quad x = 0$ or $2 - 3x = 0$

i.e. $x = 0$ or $x = \frac{2}{3}$.

(iii) From (ii), $x = 0$, $\frac{d^2y}{dx^2} = 0$,

so go back to first derivative, i.e.

x	0^-	0	0^+
$\frac{dy}{dx}$	+	0	+

$\therefore$ Horizontal point of inflexion at $(0, 7)$.

At $x = 1$, $\frac{d^2y}{dx^2} = 24 - 36$

$= -12$.

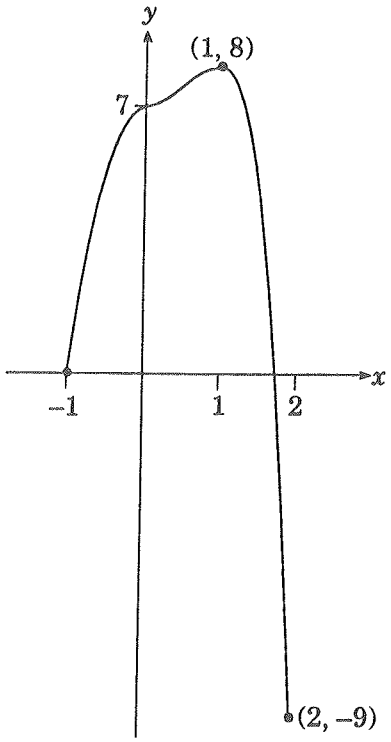
That is, relative maximum turning point at $(1, 8)$.

(iv) $x = -1$, $\quad y = 7 - 4 - 3$

$= 0$, i.e. $(-1, 0)$

$x = 2$, $\quad y = 7 + 32 - 48$

$= -9$, i.e. $(2, -9)$



QUESTION SIX

(a) $V = \pi\int_0^{10} x^2\,dy$ where $x^2 = \left(\frac{y^2}{30}\right)^2 = \frac{y^4}{900}$

$\therefore \quad V = \frac{\pi}{900}\int_0^{10} y^4\,dy$

$= \frac{\pi}{4500}\left[y^5\right]_0^{10}$

$= \frac{\pi}{4500}(100\,000 - 0)$

$= 22\frac{2}{9}\pi$

$\approx 69{\cdot}8\ \text{cm}^3$ (or 69·8 mL).

(b) (i) $y = \frac{4}{x}$ —①

$y = 5 - x$ —②

Substitute for $y = 5 - x$ from ② into ①:

$\therefore \quad 5 - x = \frac{4}{x}$

$5x - x^2 = 4$

$x^2 - 5x + 4 = 0$

$(x-1)(x-4) = 0$

$\therefore \quad x = 1$ or $x = 4$.

That is, x coordinate of A is 1, and x coordinate of B is 4.

(ii) $A = \int_1^4 (5-x)\,dx - \int_1^4 \frac{4}{x}\,dx$

$= \left[5x - \frac{1}{2}x^2\right]_1^4 - 4\left[\log_e x\right]_1^4$

$= (20-8) - \left(5 - \frac{1}{2}\right)$

$\quad - 4\left(\log_e 4 - \log_e 1\right)$

$= 7\frac{1}{2} - 4\log_e 4$

$\approx 1{\cdot}955$ units^2.

(c) (i) When $t = 0$, $R = R_0$

and when $t = 20$, $R = \frac{1}{2}R_0$

$\therefore \quad \frac{1}{2}R_0 = R_0 e^{-20k}$

$e^{-20k} = \frac{1}{2}$

$-20k = \log_e \frac{1}{2} = -\log_e 2$

$20k = \log_e 2$

$\therefore \quad k = \frac{1}{20}\log_e 2$

$\approx 0{\cdot}034\,657\ldots$

(ii) $R = 0{\cdot}3R_0$, $t = ?$

$\therefore \quad 0{\cdot}3R_0 = R_0 e^{-kt}$

$e^{-kt} = 0{\cdot}3$

$-kt = \log_e 0{\cdot}3$

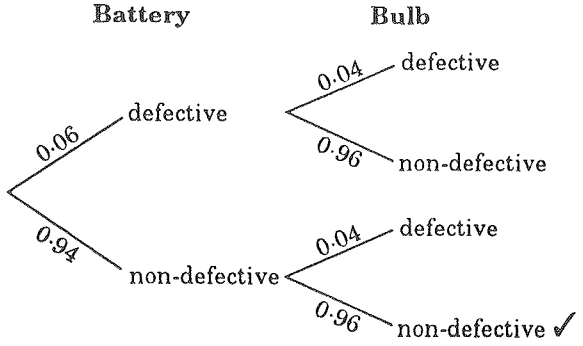
$\therefore \quad t = \frac{\log_e 0{\cdot}3}{-0{\cdot}034\,657\ldots}$

$\therefore \quad t \approx 34{\cdot}74 \quad (\approx 20 + 15).$

That is, after nearly 15 *more* years. (*Note*: Although 30% of 50 = 15, the resemblance to the answer is pure coincidence.)

QUESTION SEVEN

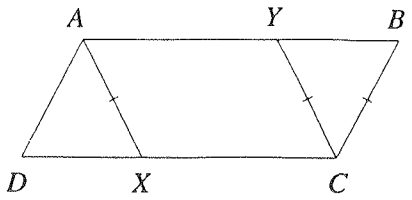
(a)



P(both non-defective) $= 0{\cdot}94 \times 0{\cdot}96$

$= 0{\cdot}9024.$

(b) (i)



(ii) $\angle ADX = \angle CBY$ because the opposite angles of a parallelogram are equal.

(iii) $AD = BC$, opposite sides of parallelogram equal.
But $AX = BC$, given.
$\therefore$ $AD = AX$.

(iv) ΔCBY is isosceles, two sides equal.
$\therefore$ $\angle CBY = \angle BYC = \theta$ (say), equal angles opposite equal sides in isosceles triangle.

$\therefore$ In ΔADX, which is also isosceles, from (iii), $\angle ADX = \theta$, and so $\angle AXD = \theta$, too.

Also, from (iii), $AD = BC$,
$\therefore$ $\Delta ADX \equiv \Delta CBY$, (**AAS**).

OR ΔADX and ΔCBY are both isosceles with base angles equal, by (ii).

$\therefore$ $\angle DAX = \angle BCY$ (angle sum of a triangle)

Hence in ΔADX and ΔCBY,

$AD = BC$ (opposite sides of parallelogram)

$AX = CY$ (given)

$\therefore$ $\angle DAX = \angle BCY$ (proved above)

$\therefore$ $\Delta ADX \equiv \Delta CBY$, (**SAS**).

(v) $BY = DX$, corresponding sides of congruent triangles.
$AB = CD$, opposite sides of parallelogram equal.
$\therefore$ $AB - BY = CD - DX$,
that is, $AY = CX$.
Also, $AX = CY$, given.
$\therefore$ $AYCX$ is a parallelogram, both pairs of opposite sides equal.

There are many alternative methods such as using the two congruent triangles to show that the opposite angles of $AYCX$ are equal; or that AY and XC are both equal and parallel.

(c) Although the level of pollutant has increased over the past 50 years, the rate of increase has slowed down to almost zero. The level of pollutant is now almost constant.

(d) (i) $\log_a\left(\frac{b}{c}\right) = \log_a b - \log_a c$

$= 2{\cdot}75 - 0{\cdot}25$

$= 2{\cdot}5.$

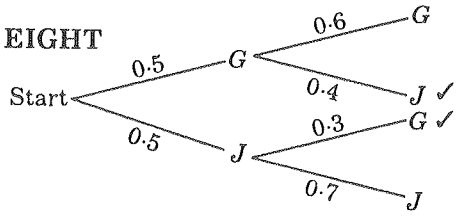
(ii) $\log_a (bc)^2 = 2(\log_a b + \log_a c)$

$= 2(2{\cdot}75 + 0{\cdot}25)$

$= 6.$

QUESTION EIGHT

(a) (i)



(ii) P(Greg wins exactly one round)

$= P(GJ) + P(JG)$

$= 0{\cdot}5 \times 0{\cdot}4 + 0{\cdot}5 \times 0{\cdot}3$

$= 0{\cdot}35.$

(b) (i) $n = 10,\ R = 1{\cdot}08,\ P = 10\,000$

$A = 10\,000 \times (1{\cdot}08)^{10}$

$\therefore$ Final amount = \$21589·25.

(ii) Final amount $= 10\,000 \times (1{\cdot}08)^{10}$
$+ 1000 \times (1{\cdot}08)^9 + 1000 \times (1{\cdot}08)^8$
$+ \cdots + 1000$

Now, $1000 + 1000 \times (1{\cdot}08)^1$
$+ \cdots + 1000 \times (1{\cdot}08)^9$

$= 1000 \times (1 + 1{\cdot}08 + \cdots + 1{\cdot}08^9)$

$= 1000 \times \frac{1(1{\cdot}08^{10} - 1)}{1{\cdot}08 - 1}$

since this is a GP with $a = 1$, $r = 1{\cdot}08,\ n = 10$

$= 14\,486{\cdot}562\,46\ldots$

$\therefore$ Final amount

$= 21589{\cdot}25 + 14\,486{\cdot}56$

$= \$36\,075{\cdot}81.$

(iii) $35\,478 = 10\,000 \times R^{10}$

$\therefore\ R^{10} = 3{\cdot}5478$

$\therefore\ R = (3{\cdot}5478)^{\frac{1}{10}}$

$= 1{\cdot}135$

$\therefore$ Annual rate of interest = 13·5%.
(Noting that $R = 1 + r$, where r is the rate of interest).

QUESTION NINE

(a) (i) Using Pythagoras' Theorem,

$w^2 + d^2 = 15^2$

$\therefore\ d^2 = 225 - w^2$

$\therefore\ S = kw(225 - w^2)$

$= k(225w - w^3).$

(ii) $\frac{dS}{dw} = k(225 - 3w^2)$

$= 0$ for max. or min.

$\therefore\ 3w^2 = 225$

$w^2 = 75$

$\therefore\ w = \pm\sqrt{75}$, i.e. $\pm 5\sqrt{3}$.

Note that $-5\sqrt{3}$ can be rejected, as $w > 0$.

$\frac{d^2S}{dw^2} = k \times -6w$

< 0 when $w = 5\sqrt{3}$.

That is, maximum strength when $w = 5\sqrt{3}$.

Now, $d^2 = 225 - w^2$.

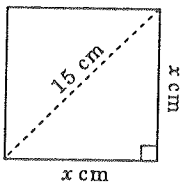
$\therefore$ When $w = 5\sqrt{3}$,

$d^2 = 150$

i.e. $d = 5\sqrt{6}$ (since $d > 0$).

$\therefore$ Dimensions are $5\sqrt{3}$ cm by $5\sqrt{6}$ cm.

(iii) For square beam, let $d = w = x$



$\therefore\ 2x^2 = 225$

$x^2 = \frac{225}{2}$

$\therefore\ S_{\text{square}} = k \times \frac{225}{2} \times \sqrt{\frac{225}{2}}$

$\approx 1193{\cdot}24k.$

Now, $S_{\text{rect.}} = k \times 150 \times 5\sqrt{3}$

$\approx 1299k.$

$\therefore\ \frac{S_{\text{rect.}}}{S_{\text{square}}} \approx \frac{1299}{1193}$

$\approx 1{\cdot}0889.$

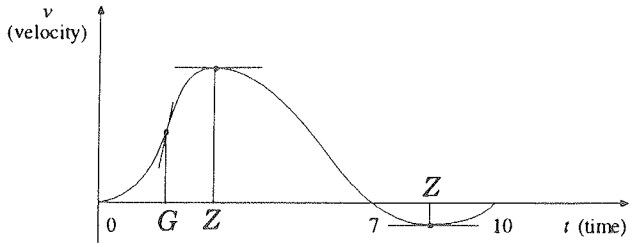
That is, more than 8% stronger.

(b) (i) Acceleration $= \frac{dv}{dt}$.

$\therefore$ zero acceleration means $\frac{dv}{dt} = 0$.

i.e., Z is the time(s) where there are stationary points on graph.

(ii) Maximum acceleration where $\frac{dv}{dt}$ is greatest.
i.e., G is the time where the gradient of the graph is steepest.



Note tangent cuts curve at G.

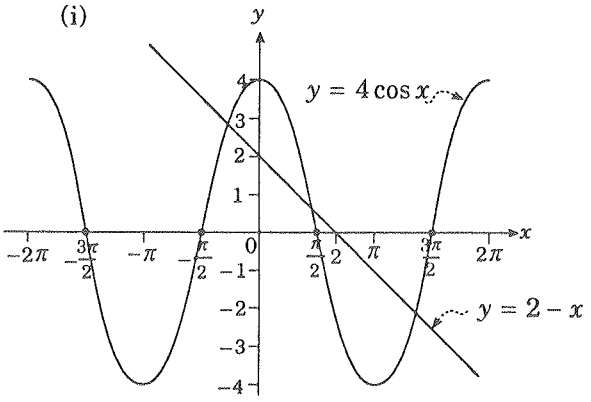
(iii) Furthest at $t = 7$.

At $t = 0$, the particle is at its initial position. The distance travelled from this initial position is given by the area under the curve, taking positive and negative areas into account. The area increases from $t = 0$ to $t = 7$, and after $t = 7$, this area decreases.

Because the area below the x axis is smaller than the area above the x axis, the particle does not return to its initial position and so the maximum distance occurs when $t = 7$.

QUESTION TEN

(a) (i)



(ii) $-\frac{\pi}{2} \approx -1{\cdot}57$

$\frac{3\pi}{2} \approx 4{\cdot}7$

From the graph, it can be clearly seen that the two graphs only meet between

$$-\frac{\pi}{2} \text{ and } \frac{3\pi}{2}.$$

Since $-2 < -\frac{\pi}{2}$ and $\frac{3\pi}{2} < 6$,

the graphs can only meet between $x = -2$ and $x = 6$. The solutions of the given equation are determined by the points of intersection, hence all solutions are between –2 and 6.

(b) (i) $x_A = -6 + 2t - \frac{1}{2}t^2$

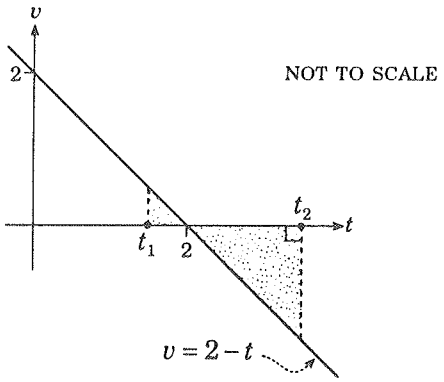
$\therefore \quad v_A = 2 - t$

$x_B = 4\sin t$

$\therefore \quad v_B = 4\cos t$

(ii) Note that $t \geq 0$.
Same velocity when $v_A = v_B$.
From (a), for $t \geq 0$, there are only two times that the graphs meet, that is at t_1 and t_2.

(iii)



Note from the graph that $v = 0$ and v changes sign at $t = 2$, which is between t_1 and t_2.

That is, the particle moves in the positive direction from t_1 to 2, and in the negative direction from 2 to t_2.

We must find these two distances separately, and add them to find the total distance travelled.

Distance travelled

$$= \int_{t_1}^{2} (2-t)\ dt + \left| \int_{2}^{t_2} (2-t)\ dt \right|$$

$$= \int_{t_1}^{2} (2-t)\ dt + \int_{t_2}^{2} (2-t)\ dt$$

$$= \left[2t - \frac{1}{2}t^2 \right]_{t_1}^{2} + \left[2t - \frac{1}{2}t^2 \right]_{t_2}^{2}$$

$$= (4-2) - \left(2t_1 - \frac{1}{2}t_1^2 \right) + (4-2) - \left(2t_2 - \frac{1}{2}t_2^2 \right)$$

$$= 4 - 2\left(t_1 + t_2 \right) + \frac{1}{2}\left(t_1^2 + t_2^2 \right).$$

OR The distance is the sum of the two shaded triangles. The gradient of $v_A = 2 - t$ is -1, hence each triangle is right-angled, isosceles.

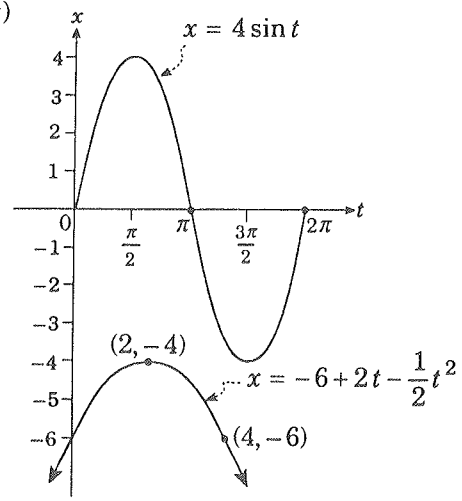
$\therefore$ Distance

$$= \frac{1}{2}\left(2 - t_1 \right)^2 + \frac{1}{2}\left(t_2 - 2 \right)^2$$

$$= \frac{1}{2}\left(4 - 4t_1 + t_1^2 + t_2^2 - 4t_2 + 4 \right)$$

$$= 4 - 2\left(t_1 + t_2 \right) + \frac{1}{2}\left(t_1^2 + t_2^2 \right).$$

(iv)



$$x = -6 + 2t - \frac{1}{2}t^2$$

$$\frac{dx}{dt} = 2 - t$$

$= 0$ when $t = 2$.

$t = 2$, $x = -6 + 4 - 2$

$= -4$.

$x = -6 + 2t - \frac{1}{2}t^2$ is concave down, so has its *maximum* turning point at $(2, -4)$.

$x = 4\sin t$ has a *minimum* turning point at $\left(\frac{3\pi}{2}, -4 \right)$.

Graphically it can be seen that the two particles never meet.