1994 HIGHER SCHOOL CERTIFICATE SOLUTIONS

2/3 UNIT (COMMON) MATHEMATICS

QUESTION ONE

(a) $\dfrac{4{\cdot}23}{\sqrt{6{\cdot}14-1{\cdot}78}} = 2{\cdot}0258$

$= 2{\cdot}03$ correct to 2 decimal places.

(b) $2-3(x-5) = 2-3x+15$

$= 17-3x.$

(c) $\dfrac{d}{dx}(3x^5-4x+7) = 15x^4-4$

(d) $|x-1| = 4$

$(x-1) = 4$ or $-(x-1) = 4$

$x-1 = 4$ $\qquad -x+1 = 4$

$x = 5$ $\qquad -x = 3$

$x = -3$

$\therefore\ x = 5$ or $-3.$

(e) $149\,492\,000$ km $= 1{\cdot}495\times10^8$ correct to 4 significant figures.

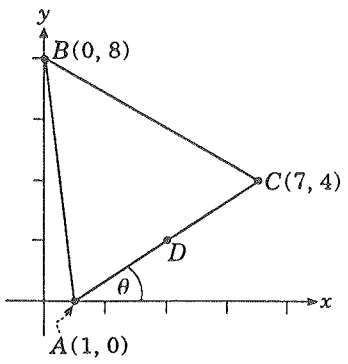
(f) $A_n = P\left(1+\dfrac{r}{100}\right)^n$ where $n = 5\times4 = 20$

$P = 1000$

$A_{20} = 1000\left(1+\dfrac{2}{100}\right)^{20}$ $\qquad r = \dfrac{8}{4} = 2\%$

$= \$1485{\cdot}95.$

QUESTION TWO



(a) Gradient $= \dfrac{\text{rise}}{\text{run}} = \dfrac{4}{7-1}$

$= \dfrac{4}{6}$

$= \dfrac{2}{3}$ is the gradient of AC.

(b) $\tan\theta = \dfrac{2}{3}$

$\theta = 33^\circ\ 41'.$

(c) Line AC, $m = \dfrac{2}{3}$, $A(1, 0)$

$y-y_1 = m(x-x_1)$

$y-0 = \dfrac{2}{3}(x-1)$

$y = \dfrac{2}{3}x-\dfrac{2}{3}$

$3y = 2x-2$

$\therefore\ 2x-3y-2 = 0$ is the equation of line AC.

(d) Midpoint of AC

$x = \dfrac{x_1+x_2}{2},\ y = \dfrac{y_1+y_2}{2}$

$x = \dfrac{1+7}{2},\quad y = \dfrac{0+4}{2}$

$x = 4,\quad y = 2$

$\therefore$ The point D is $(4, 2)$.

(e) Gradient $AC = \dfrac{2}{3}$

Gradient $BD = \dfrac{\text{rise}}{\text{run}} = \dfrac{8-2}{0-4}$

$= \dfrac{6}{-4}$

$= -\dfrac{3}{2}.$

Gradient $AC \times$ gradient $BD = \dfrac{2}{3}\times-\dfrac{3}{2}$

$= -1.$

$\therefore$ BD is perpendicular to AC.

(f) ΔABC is isosceles (because BD is perpendicular bisector of AB).

(g) Distance $AC = \sqrt{(7-1)^2+(4-0)^2}$

$= \sqrt{36+16}$

$= \sqrt{52}$.

Distance $BD = \sqrt{(4-0)^2+(2-8)^2}$

$= \sqrt{16+36}$

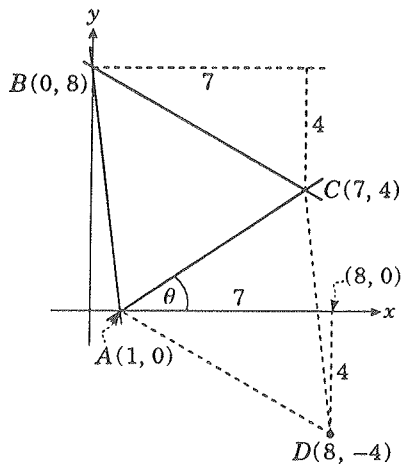
$= \sqrt{52}$.

Area of $\Delta ABC = \frac{1}{2}bh$

$= \frac{1}{2}\times\sqrt{52}\times\sqrt{52}$

$= 26$ square units.

(h)



NOT TO SCALE

D is point $(8,-4)$.

QUESTION THREE

(a) (i) $\frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{d}{dx}\left(x^{-4}\right)$

$= -4x^{-5}$

$= \frac{-4}{x^5}$.

(ii) $\frac{d}{dx}\cos(x^2) = -\sin(x^2)\cdot 2x$

$= -2x\sin(x^2)$.

(iii) $\frac{d}{dx}x\tan x = x\frac{d}{dx}(\tan x)+\tan x\frac{d}{dx}(x)$

$= x\cdot\sec^2 x+\tan x\cdot 1$

$= x\sec^2 x+\tan x$.

(b) (i) To prove $\Delta JMN \parallel\!\parallel \Delta JKL$

$\angle MJN = \angle KJL$ (common)

$\angle JMN = \angle JKL$ (corresponding, $MN \parallel KL$)

$\therefore \Delta JMN \parallel\!\parallel \Delta JKL$ as they are equiangular.

(ii) $\frac{MK}{JK} = \frac{NL}{JL}$

$\frac{MK}{8} = \frac{3}{12}$

$12MK = 24$

$MK = 2$ cm.

(c) (i) $\int e^{2x}\,dx = \frac{e^{2x}}{2}+C$

$= \frac{1}{2}e^{2x}+C$.

(ii) $\int_1^9\left(1+\sqrt{x}\right)dx$

$= \int_1^9\left(1+x^{\frac{1}{2}}\right)dx$

$= \left[x+\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^9 = \left[x+\frac{2\sqrt{x^3}}{3}\right]_1^9$

$= \left(9+\frac{2\sqrt{9^3}}{3}\right)-\left(1+\frac{2\sqrt{1^3}}{3}\right)$

$= 9+18-1-\frac{2}{3}$

$= 25\frac{1}{3}$.

QUESTION FOUR

(a) By the cosine rule:

$x^2 = a^2+b^2-2ab\cos C$

$x^2 = 350^2+200^2-2\times 350\times 200\cos 10°$

$x^2 = 24\,626{\cdot}9$

$x = 156{\cdot}9$ m (correct to 1 decimal place)

$\therefore$ It is 156·9 m from B to H.

(b) $120° = \frac{120\pi}{180}$ radians

$= \frac{2\pi}{3}$ radians

Area of sector $RS = \frac{1}{2}\times 50^2\times\frac{2\pi}{3}$

$= 2617{\cdot}99$.

Area of sector $UT = \frac{1}{2}\times 20^2\times\frac{2\pi}{3}$

$= 418{\cdot}88$.

Area of $RSTU$ = 2617·99 − 418·88

= 2199·11 cm^2

(correct to 2 decimal places).

(c) $\displaystyle\int_0^1 xe^{x^2}\,dx = \frac{1}{2}\int_0^1 2xe^{x^2}\,dx$

$= \dfrac{1}{2}\Big[\,e^{x^2}\,\Big]_0^1$

$= \dfrac{1}{2}\left(e^1 - e^0\right)$

$= \dfrac{1}{2}(e-1).$

(d) 7, 14, 21, ...

AP with $a = 7$, $d = 7$.

(i) $T_n = a + (n-1)d$

Let $T_n = 1000$

$\therefore \quad 1000 = 7 + (n-1)7$

$1000 = 7 + 7n - 7$

$7n = 1000$

$n = 142{\cdot}857,$

$\therefore \quad n = 142.$

$\therefore$ The largest multiple of 7 less than 1000 is $142 \times 7 = 994$.

(ii) $S_n = \dfrac{n}{2}(a + \ell)$

$S_{142} = \dfrac{142}{2}(7 + 994)$

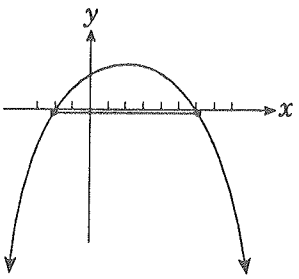
$= 71\,071.$

QUESTION FIVE

(a) $12 + 4m - m^2 > 0$

Let $y = 12 + 4m - m^2$

$y = (6-m)(2+m).$



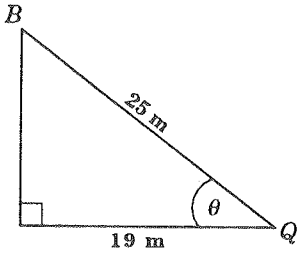
Parabola is concave downwards and cuts x axis at 6 and −2.

$\therefore \quad 12 + 4m - m^2 > 0$

for $-2 < m < 6.$

(b) $\cos\theta = \dfrac{19}{25}$

$\theta = 40°\,32'.$



(c) (i) $x = 6 - 2t + 8\ln(t+3)$

At $t = 0$,

$x = 6 - 2(0) + 8\ln(3)$

= 14·79 m (correct to 2 decimal places).

(ii) $x = 6 - 2t + 8\ln(t+3)$

$\dfrac{dx}{dt} = v = -2 + \dfrac{8}{t+3}$

$\dfrac{d^2x}{dt^2} = a = -8(t+3)^{-2}$

$= -\dfrac{8}{(t+3)^2}.$

(iii) $v = -2 + \dfrac{8}{t+3}$

At $v = 0 \quad = -2 + \dfrac{8}{t+3}$

$0 = -2(t+3) + 8$

$0 = -2t - 6 + 8$

$= -2t + 2$

$2t = 2$

$t = 1$ second when the velocity is 0.

(d) $V = \pi\displaystyle\int_0^9 \left(6\sqrt{x}\right)^2 dx$

$= \pi\displaystyle\int_0^9 36x\,dx$

$= \pi\left[\dfrac{36x^2}{2}\right]_0^9$

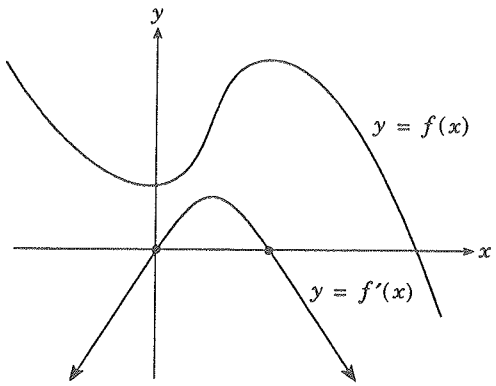
$= \pi\left(\dfrac{36(9)^2}{2} - 0\right)$

$\therefore$ Volume is 1458π units^3.

QUESTION SIX

(a) This statement is only true if there is an equal number of each of the green, black, and red jellybeans.

(b)



(c) $y = x^3 - 6x + 4$

$\dfrac{dy}{dx} = 3x^2 - 6$

$\dfrac{d^2y}{dx^2} = 6x.$

(i) For stationary points, $\frac{dy}{dx} = 0$,

$\therefore \quad 3x^2 - 6 = 0$

$3x^2 = 6$

$x^2 = 2$

$x = \pm\sqrt{2}$.

At $x = +\sqrt{2}$

$y = -1{\cdot}7$

$\frac{d^2y}{dx^2} > 0$,

$\therefore$ local minimum at $\left(\sqrt{2}, -1{\cdot}7\right)$.

At $x = -\sqrt{2}$

$y = 9{\cdot}7$

$\frac{d^2y}{dx^2} < 0$,

$\therefore$ local maximum at $\left(-\sqrt{2}, 9{\cdot}7\right)$.

(ii) For point of inflexion, $\frac{d^2y}{dx^2} = 0$,

$\therefore \quad 6x = 0$

$x = 0$

$y = 4$.

TEST:

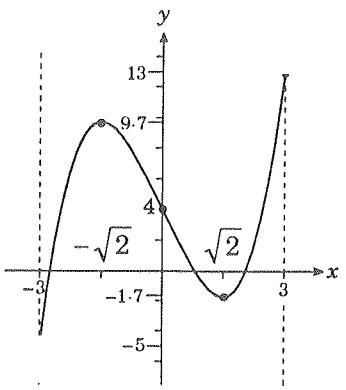
When x is a little less than 0, $\frac{d^2y}{dx^2} < 0$.

When x is a little more than 0, $\frac{d^2y}{dx^2} > 0$.

$\therefore$ A point of inflexion occurs at (0, 4).

(iii) At $x = -3$, $y = -5$,
at $x = 3$, $y = 13$.

(iv) Maximum value of $x^3 - 6x + 4$ in $-3 \le x \le 3$ is 13 when $x = 3$.



QUESTION SEVEN

(a) (i) $V = \pi r^2 h$

$500 = \pi r^2 h$

$h = \frac{500}{\pi r^2}$.

(ii) $S = 2\pi r^2 + 2\pi rh$

$S = 2\pi r^2 + 2\pi r\left(\frac{500}{\pi r^2}\right)$

$= 2\pi r^2 + \frac{1000}{r}$.

(iii) $S = 2\pi r^2 + \frac{1000}{r}$

$\frac{dS}{dr} = 4\pi r + \frac{-1000}{r^2}$

$\frac{dS}{dr} = 0, \quad 0 = 4\pi r + \frac{-1000}{r^2}$

$0 = 4\pi r^3 - 1000$

$4\pi r^3 = 1000$

$\pi r^3 = 250$

$r^3 = \frac{250}{\pi}$

$r = \sqrt[3]{\frac{250}{\pi}}$

$\approx 4{\cdot}3$ cm.

CHECK:

$\frac{d^2S}{dr^2} = 4\pi + \frac{2000}{r^3}$

> 0 if $r = 4{\cdot}3$

$\therefore$ area is a minimum.

(b) $2\log_5 3 = \log_5 x - \log_5 6$

$\log_5 x = 2\log_5 3 + \log_5 6$

$\log_5 x = \log_5 3^2 + \log_5 6$

$\log_5 x = \log_5 9 + \log_5 6$

$\log_5 x = \log_5 54$

$\therefore \quad x = 54$.

(c) $T = T_0 e^{-kt}$

(i) – At $t = 0$, $T = 10\,000$

$10\,000 = T_0 e^{-k(0)}$

$= T_0 e^0$

$\therefore \quad T_0 = 10\,000$.

At $t = 1$, $\quad T = 9000$

$9000 = 10\,000 e^{-k(1)}$

$e^{-k} = \frac{9000}{10\,000}$

$$e^{-k} = \frac{9}{10}$$

$$\ln e^{-k} = \ln \frac{9}{10}$$

$$-k = \ln \frac{9}{10}$$

$$k = -\ln \frac{9}{10}.$$

(ii) $40 = 10\,000 e^{\ln \frac{9}{10} t}$

$$e^{\ln \frac{9}{10} t} = \frac{1}{250}$$

$$\ln e^{\ln \frac{9}{10} t} = \ln \frac{1}{250}$$

$$\ln \frac{9}{10} t = \ln \frac{1}{250}$$

$$t = \frac{\ln \frac{1}{250}}{\ln \frac{9}{10}}$$

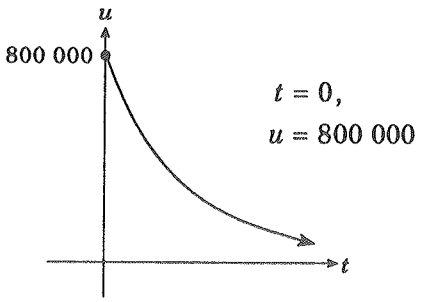
$t = 52{\cdot}4$ years

≈ 53 years.

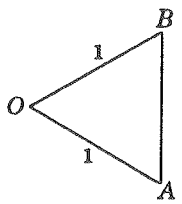
QUESTION EIGHT

(a) (i) If $\frac{du}{dt} < 0$, then the unemployment rate must be decreasing over that period.

(ii) If $\frac{d^2u}{dt^2} > 0$, then the function is concave up.



(b) (i) $AO = BO = 1$ cm



$$\angle AOB = \frac{360}{6}$$

$$= 60°.$$

$$AB^2 = 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos 60°$$

by the cosine rule.

$$AB^2 = 1 + 1 - 2 \times \cos 60°$$

$$= 1$$

$$\therefore \quad AB = 1.$$

$\therefore$ ΔOAB is equilateral (all sides = 1 cm).

OR $\quad \angle AOB = \frac{1}{6}$ of $360°$

$$= 60°.$$

$$OA = OB$$

$$\therefore \quad \angle OAB = \angle OBA$$

$$\therefore \text{ each angle } = \frac{1}{2}(180° - 60°)$$

$$= 60°.$$

All angles are 60°, so triangle is equilateral.

(ii) Area of $\Delta OAB = \frac{1}{2} ab \sin C$

$$= \frac{1}{2} \times 1 \times 1 \times \sin 60°$$

$$= \frac{1}{2} \times 1 \times \frac{\sqrt{3}}{2}$$

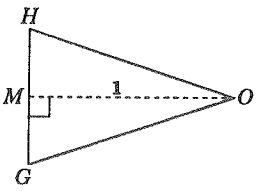
$$= \frac{\sqrt{3}}{4} \text{cm}^2.$$

$$\therefore \text{ Area of hexagon } = 6 \times \frac{\sqrt{3}}{4}$$

$$= \frac{3\sqrt{3}}{2} \text{cm}^2.$$

(iii) As $OH = OG$,

ΔOGH is isosceles.



$$\therefore \quad \angle MOG = 30°$$

$$\tan 30 = \frac{MG}{OM}$$

$$\tan 30 = \frac{MG}{1}$$

$$MG = \tan 30 = \frac{1}{\sqrt{3}}$$

$$\therefore \quad HG = 2 \times \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}.$$

$$\therefore \quad \text{Area } \Delta OGH = \frac{1}{2} bh$$

$$= \frac{1}{2} \times \frac{2}{\sqrt{3}} \times 1$$

$$= \frac{1}{\sqrt{3}} \text{ cm}^2.$$

$\therefore$ Area of outer hexagon

$$= 6 \times \sqrt{3}$$

$$= \frac{6}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{6\sqrt{3}}{3}$$

$$= 2\sqrt{3} \text{ cm}^2.$$

(iv) Area of circle $= \pi r^2$
$= \pi$, as $r = 1$ cm.

But the area of the circle lies between the area of the inner hexagon and the area of the outer hexagon.

$\therefore$ Area inner hexagon < area of circle < area outer hexagon,

$$\therefore \quad \frac{3\sqrt{3}}{2} < \pi < 2\sqrt{3}.$$

Note Approximating the area of a circle by calculating the areas of inner and outer regular hexagons is called the method of exhaustion. It was first used in this way by Archimedes.

QUESTION NINE

(a) (i) Solve simultaneously.

$$y = x+2, \quad y = x^2-5x+11$$
$$\therefore \quad x+2 = x^2-5x+11$$
$$0 = x^2-6x+9$$
$$(x-3)(x-3) = 0$$
$$\therefore \quad x = 3.$$

As there is only one point of intersection, $y = x+2$ must be a tangent to $y = x^2-5x+11$.

(ii) At $x = 3$, $y = 5$, $\therefore$ $Q(3, 5)$

$$y = x^2-5x+11$$
$$\frac{dy}{dx} = 2x-5$$

When $x = 3$, $\dfrac{dy}{dx} = 1$.

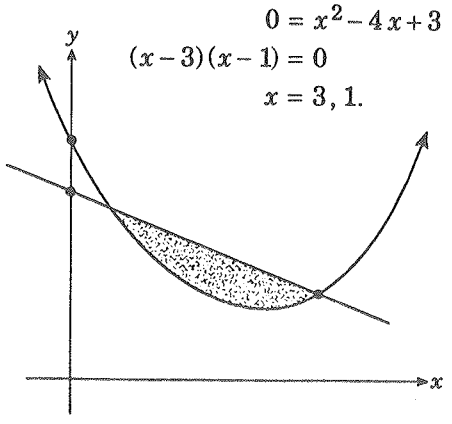
$\therefore$ Gradient of tangent to curve is 1.
Gradient of normal to curve is -1.

$$y-y_1 = m(x-x_1)$$
$$y-5 = -1(x-3)$$

$\therefore$ $y = -x+8$. Normal to curve at $(3, 5)$.

(iii) $y = -x+8$, $y = x^2-5x+11$

$$-x+8 = x^2-5x+11$$
$$0 = x^2-4x+3$$
$$(x-3)(x-1) = 0$$
$$x = 3, 1.$$



$$A = \int_1^3 \Big((-x+8)-(x^2-5x+11)\Big)\,dx$$
$$= \int_1^3 \Big((-x^2+4x-3)\Big)\,dx$$
$$= \left[\frac{-x^3}{3}+2x^2-3x\right]_1^3$$
$$= (-9+18-9)-\left(-\frac{1}{3}+2-3\right)$$
$$= 1\frac{1}{3} \text{ square units.}$$

(b) Distance from $P(x, y)$ to $y-3 = 0$ is

$$d_1 = \frac{|0+y-3|}{\sqrt{0+1^2}}$$
$$= |y-3| \text{ as } y < 3 \text{ in this region}$$
$$= 3-y.$$

Distance from $P(x, y)$ to $3x+4y-18 = 0$ is

$$d_2 = \frac{|3x+4y-18|}{\sqrt{9+16}}$$
$$= \frac{3x+4y-18}{5}.$$

As these two distances are equidistant, $d_1 = d_2$.

$$\therefore \quad 3-y = \frac{3x+4y-18}{5}$$
$$15-5y = 3x+4y-18$$
$$3x+9y-33 = 0$$

or $\quad x+3y-11 = 0.$

So locus lies on this line.
But to be in the shaded region it is only the part of the line to the right of the point where the line $3x+4y-18 = 0$ meets the line $y = 3$.

At $y = 3$, $3x+12-18 = 0$
$$x = 2$$

$\therefore$ locus is $x+3y-11 = 0$ for $x \geq 2$.

QUESTION TEN

(a) (i) 6, 3; 3, 6; 4, 5; 5, 4; and 36 possible outcomes,

$$\therefore \quad P(\text{win}) = \frac{4}{36} = \frac{1}{9}.$$

(ii) $$1-\left(\frac{1}{6}+\frac{1}{9}\right) = \frac{13}{18}.$$

(iii) $$P(W_1) = \frac{1}{9}, \quad P(W_2) = \frac{1}{9}\times\frac{13}{18},$$
$$P(W_3) = \frac{1}{9}\times\left(\frac{13}{18}\right)^2$$

$$\therefore \text{ Probability } = \frac{1}{9}\times\frac{1}{9}\times\frac{13}{18}+\frac{1}{9}\times\left(\frac{13}{18}\right)^2.$$

(iv) S_∞ where $a=\frac{1}{9}$, $r=\frac{13}{18}$,

$$\therefore P(W)=\frac{\frac{1}{9}}{1-\frac{13}{18}}=\frac{2}{5}.$$

Alternative solution (1): $P(\text{win})=\frac{1}{9}$

$P(\text{lose})=\frac{1}{6}$

$$\therefore P(\text{eventually wins})=\frac{P(\text{win})}{P(\text{win})+P(\text{lose})}$$

$$=\frac{\frac{1}{9}}{\frac{1}{9}+\frac{1}{6}}$$

$$=\frac{4}{10}\quad=\frac{2}{5}.$$

Alternative solution (2):

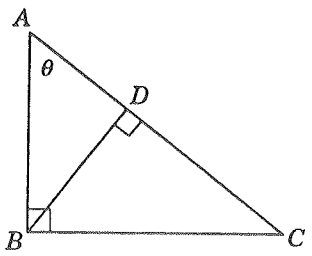
No. of ways to win = 4

No. of ways to lose = 6

$$P(\text{win})=\frac{\text{no. ways to win}}{\text{no. ways to win}+\text{no. ways to lose}}$$

$$=\frac{4}{4+6}\quad=\frac{4}{10}\quad=\frac{2}{5}.$$

(b) (i)



(ii) $\cos\theta=\frac{AD}{AB}$

$\therefore \quad AD=AB\cos\theta$

$\tan\theta=\frac{BC}{AB}$

$\therefore \quad BC=AB\tan\theta$

$\sec\theta=\frac{AC}{AB}$

$\therefore \quad AC=AB\sec\theta.$

Since $6AD+BC=5AC,$

$6AB\cos\theta+AB\tan\theta=5AB\sec\theta$

$\therefore \quad 6\cos\theta+\tan\theta=5\sec\theta.$

(iii) $6\cos\theta+\frac{\sin\theta}{\cos\theta}=\frac{5}{\cos\theta}$

$6\cos^2\theta+\sin\theta=5$

$\therefore \quad 6(1-\sin^2\theta)+\sin\theta=5$

$6\sin^2\theta-\sin\theta-1=0.$

(iv) $(2\sin\theta-1)(3\sin\theta+1)=0$

$\sin\theta=\frac{1}{2}$ or $-\frac{1}{3}$

and $0°\le\theta\le 90°$ because θ is a right-angled triangle,

$\therefore \quad \theta=30°.$