1993 HIGHER SCHOOL CERTIFICATE SOLUTIONS

2/3 UNIT MATHEMATICS

QUESTION ONE

(a) $4\pi\sqrt{\dfrac{a}{g}} = 4\pi\sqrt{\dfrac{4\cdot1}{9\cdot8}}$

$= 8\cdot128\,09\cdots$

$\approx 8\cdot1$ (2 sig. fig.).

(b) Since (3, 2) lies on $6x - ky = 4$, substitute $x = 3$ and $y = 2$

$\therefore \quad 6\times3 - k\times2 = 4$

$-2k = 4 - 18$

$-2k = -14$

$k = 7.$

(c) Using Pythagoras' Theorem:

$(10-x)^2 = x^2 + 5^2$

$100 - 20x + x^2 = x^2 + 25$

$\therefore \quad -20x = 25 - 100$

$-20x = -75$

$x = 3\cdot75.$

(d) Number of routes $= 7 \times 5 = 35.$

(e) $2x + y = 7$ —①

$x - 2y = 1$ —②

From ②, $x = 1 + 2y$

Substitute in ①:

$2(1+2y) + y = 7$

$2 + 4y + y = 7$

$5y = 5$

$y = 1$

$\therefore \quad x = 1 + 2\times1$

$= 3$

$\therefore$ Solution is $x = 3,\ y = 1.$

(f) $5 - 3x < 7$

$\therefore \quad 5 < 7 + 3x$ OR $5 - 3x < 7$

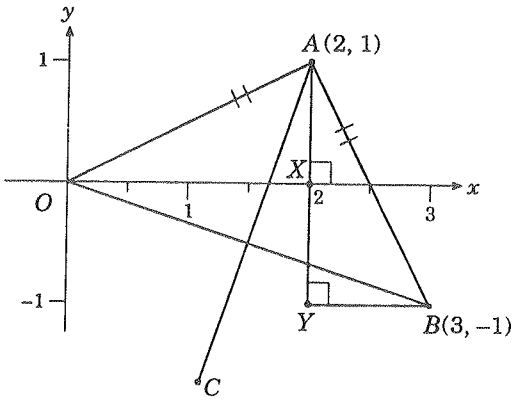
$3x > 5 - 7$ $\qquad -3x < 7 - 5$

$3x > -2$ $\qquad -3x < 2$

$x > -\dfrac{2}{3}$ $\qquad x > -\dfrac{2}{3}.$

QUESTION TWO

(a) (i)



(ii) Gradient $m_1 = \dfrac{1}{2}$, gradient $m_2 = -\dfrac{2}{1} = -2.$

(iii) If $m_1 \times m_2 = -1$, then lines are perpendicular.

Since $\dfrac{1}{2}\times-2 = -1$, then $OA \perp AB.$

(iv) $OA^2 = 2^2 + 1^2$

$= 5$

$\therefore \quad OA = \sqrt{5}$

and $\quad AB^2 = 2^2 + 1^2$

$\therefore \quad AB = \sqrt{5}$

$OA = AB.$

OR Let $X = (2,\ 0),\ Y = (2,\ -1)$ (as shown on diagram)

$OX = AY = 2$

$\hat{X} = \hat{Y} = 90°$

$AX = YB = 1$

$\therefore \quad \Delta OAX \equiv \Delta AYB$ (SAS)

$\therefore \quad OA = AB$ (corresponding sides of congruent triangles).

(v) Midpoint of OB, D is $\left(\dfrac{3}{2},\ -\dfrac{1}{2}\right).$

(vi) Let C be point $(x_1,\ y_1)$, then since $D\left(1\dfrac{1}{2},\ -\dfrac{1}{2}\right)$ is midpoint of (2, 1) and $(x_1,\ y_1)$.

$$\frac{2+x_1}{2}=\frac{3}{2} \text{ and } \frac{1+y_1}{2}=-\frac{1}{2}$$
$$\therefore\ 2+x_1=3 \qquad 1+y_1=-1$$
$$x_2=1 \qquad y_1=-2$$
$\therefore$ C is $(1,-2)$.

(vii) The diagonals OB and AC bisect each other, $\therefore$ $OABC$ is a parallelogram.
$OA=AB$, $\therefore$ $OABC$ is a rhombus.
$O\hat{A}B=90°$, $\therefore$ $OABC$ is a square.

(b) Ratio of tagged fish to number in the sample is 5 : 42. Hence, if this ratio is constant for the whole dam, then
tagged : sample number
$=5 : 42$
$=75 : N$, where N is total number
$$\therefore \frac{N}{75}=\frac{42}{5}$$
$N=630$.

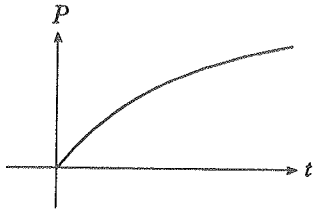
QUESTION THREE

(a) (i) $y=\sin(3x+1)$
$$\frac{dy}{dx}=3\cos(3x+1)$$

(ii) $y=\dfrac{1}{x^3}$
$=x^{-3}$
$$\frac{dy}{dx}=-3x^{-4}$$
$$=-\frac{3}{x^4}.$$

(iii) $y=x^2\ln x$
$$\frac{dy}{dx}=\ln x\times\frac{d}{dx}x^2+x^2\times\frac{d}{dx}\ln x$$
$$=2x\ln x+x^2\times\frac{1}{x}$$
$$=2x\ln x+x$$
$$=x(2\ln x+1).$$

(b)



P is the amount of pollution, t is time.

The rate at which P is increasing is expressed by $\dfrac{dP}{dt}>0$.

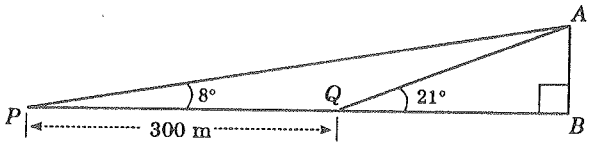
The policy to reduce pollution taking effect means that even though pollution is increasing, the rate of increase is being slowed, that is, $\dfrac{d^2P}{dt^2}<0$.

(c) (i) $\displaystyle\int(x-1)^3\,dx=\frac{(x-1)^4}{4}+c$

(ii) $\displaystyle\int_0^{\frac{\pi}{2}}\sin 2x\,dx=\left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$
$$=-\frac{1}{2}\cos\pi-\left(-\frac{1}{2}\cos 0\right)$$
$$=\frac{1}{2}+\frac{1}{2}$$
$$=1.$$

QUESTION FOUR

(a)



(i) $\angle PAQ=21°-8°=13°$.

(ii) $\dfrac{AQ}{\sin\angle APQ}=\dfrac{PQ}{\sin\angle PAQ}$ (Sine Rule, ΔPAQ)
$$\therefore \frac{AQ}{\sin 8°}=\frac{300}{\sin 13°}$$
$$\therefore AQ=185{\cdot}604\,51\cdots$$
$$\therefore AQ\approx 185{\cdot}6 \text{ m}.$$

(iii) In ΔABQ, $\dfrac{AB}{AQ}=\sin 21°$
$$\therefore AB=185{\cdot}6045\times\sin 21°$$
$$=66{\cdot}514\,707\cdots$$
$$\approx 66{\cdot}5 \text{ m}.$$

(b) (i) $N(t)=Ae^{0{\cdot}15t}$
$$\frac{dN}{dt}=A\times 0{\cdot}15e^{0{\cdot}15t}$$
$$=0{\cdot}15Ae^{0{\cdot}15t}$$
$$=0{\cdot}15N.$$
$\therefore$ Number of algae increases at a rate proportional to the number present.

(ii) When $t=3$,
$$1{\cdot}7\times 10^8=Ae^{0{\cdot}15\times 3}$$
$$\therefore A=\frac{1{\cdot}7\times 10^8}{e^{0{\cdot}45}}$$
$$=1{\cdot}083\ 968\times 10^8.$$

(iii) When $t=x+3$, $N=3{\cdot}4\times 10^8$ if number has doubled,

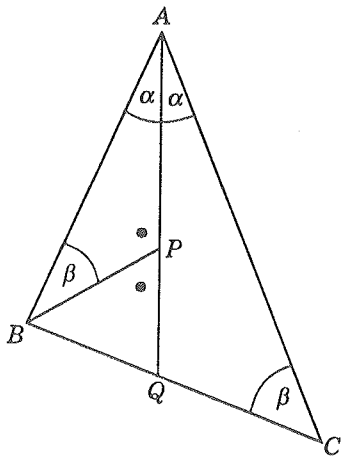
$\therefore$ with $A=1{\cdot}083\,968\times 10^8$,
$N=Ae^{0{\cdot}15t}$

$$3{\cdot}4\times10^8 = 1{\cdot}083\,968\times10^8\, e^{0{\cdot}15(x+3)}$$
$$\therefore\quad e^{0{\cdot}15(x+3)} = 3{\cdot}136\,624$$
$$\therefore\quad 0{\cdot}15(x+3) = \ln 3{\cdot}136\,625$$
$$x+3 = 7{\cdot}620\,98\cdots$$
$$x = 4{\cdot}6209\cdots$$

OR $\quad t=0 \quad N=A$
$\quad t=x \quad N=2A$

$$2A = Ae^{0{\cdot}15x}$$
$$2 = e^{0{\cdot}15x}$$
$$\ln 2 = \ln e^{0{\cdot}15x}$$
$$= 0{\cdot}15x$$
$$\therefore\quad x = \frac{\ln 2}{0{\cdot}15}$$
$$= 4{\cdot}620\,98\cdots$$

(c) (i)



(ii) Aim: To prove $BP = BQ$.
Proof:

$\angle BPQ = \angle BAP + \angle ABP$ (ext. angle of ΔABP)
$= \alpha + \beta$
$\angle PQB = \angle QAC + \angle ACQ$ (ext. angle of ΔAQC)
$= \alpha + \beta$
$\therefore\ \angle BPQ = \angle PQB$
$\therefore\ \Delta BPQ$ is isosceles
$\therefore\quad BP = BQ$ (sides opposite equal angles in ΔBPQ).

QUESTION FIVE

(a) $\quad y = x^3 - 3x + 1$

$$\frac{dy}{dx} = 3x^2 - 3$$

At the point $(-2, -1)$, $\frac{dy}{dx} = 3\times4 - 3 = 9$

$\therefore$ tangent must have gradient $m = 9$.
Now the tangent at $(-2, -1)$ with $m = 9$ is

$$y - y_1 = m(x - x_1)$$
$$y + 1 = 9(x+2)$$
$$y = 9x + 17.$$

Hence $m = 9$, $b = 17$.

(b) (i) $\int_0^8 f(x)\ dx$ represents the signed areas bounded by the function and the x axis.
Since the areas of the two quarter-circles are equal and of opposite sign, these do not contribute to the value. We require the signed areas of the triangle with hypotenuse BC, and the square side 2 units,

$$\therefore\ \int_0^8 f(x)\ dx = -\left(\frac{1}{2}\times2\times2\right) - 2\times2$$
$$= -6.$$

(ii) The function is not differentiable at any point where the curve is not smooth or not continuous. Hence it is not differentiable at $x = 2$ and $x = 4$ (the end points $x = 0$ and $x = 8$ are not included, and at $x = 6$, the gradient is continuous).

(c) (i) 1st: $\quad 4$
2nd: $\quad 4+6$
3rd: $\quad 4+6+6$
$\quad = 4+2\times6$
kth $\quad = 4+(k-1)\times6$
$\quad = 4+6k-6$
$\quad = 6k-2.$

(ii) To fill all n troughs, we need the sum of n terms of the sequence.

$$S_n = \frac{n}{2}\left(2a+(n-1)d\right) \quad \text{where } a=4,\ d=6$$
$$= \frac{n}{2}\left(8a+(n-1)6\right)$$
$$= \frac{n}{2}(6n+2)$$
$$= n(3n+1).$$

(iii) Given $S_n = 1120$,
then $\quad n(3n+1) = 1220$
$$3n^2 + n - 1220 = 0$$
$$\therefore\quad n = \frac{-1\pm\sqrt{1+4\times3\times1220}}{6}$$
$$= \frac{-1\pm121}{6}$$
$$= \frac{120}{6} \text{ or } -\frac{122}{6}$$
$$= 20 \ (n \text{ is a positive integer}).$$

$\therefore$ There are 20 water troughs.

QUESTION SIX

(a) $\quad y = \frac{1}{4}x^4 - x^3$

$$\frac{dy}{dx} = x^3 - 3x^2 = 0 \quad \text{when } x^2(x-3) = 0$$
$$\therefore\quad x = 0,\ 3.$$

$$\frac{d^2y}{dx^2} = 3x^2 - 6x.$$

(i) When $x = 0$, $y = 0$, and $\dfrac{d^2y}{dx^2} = 0$.

When $x = 3$, $y = -6\dfrac{3}{4}$ and $\dfrac{d^2y}{dx^2} = 9$,

i.e. a possible inflexion at (0, 0) and

a minimum turning point at $\left(3,\ -6\dfrac{3}{4}\right)$.

(ii) To confirm the inflexion at (0, 0), test values of x on either side.

When $x = -0{\cdot}1$,

$$\frac{d^2y}{dx^2} = 3\times(-0{\cdot}1)^2 - 6\times -0{\cdot}1$$
$$= 0{\cdot}63.$$

When $x = 0{\cdot}1$,

$$\frac{d^2y}{dx^2} = 3\times -0{\cdot}1^2 - 6\times 0{\cdot}1$$
$$= -0{\cdot}57.$$

Hence, since the concavity changes as x moves through $x = 0$, there is an inflexion at (0, 0).

To check for other points of inflexion:

$$\frac{d^2y}{dx^2} = 3x^2 - 6x$$
$$= 0 \quad \text{when } 3x(x-2) = 0$$
$$\therefore\ x = 0,\ x = 2.$$

Testing for the inflexion at $x = 2$,

consider $x = 1{\cdot}9$,

$$\frac{d^2y}{dx^2} = 3\times 1{\cdot}9^2 - 6\times 1{\cdot}9$$
$$= -0{\cdot}57.$$

When $x = 2{\cdot}1$,

$$\frac{d^2y}{dx^2} = 3\times 2{\cdot}1^2 - 6\times 2{\cdot}1$$
$$= 0{\cdot}63.$$

Since concavity changes, there is an inflexion when $x = 2$,

and when $x = 2$, $y = \dfrac{1}{4}\times 2^4 - 2^3$

$$= -4$$

i.e. at (2, −4)

(iii) $x = -1{\cdot}5$, $y = \dfrac{1}{4}(-1{\cdot}5)^4 - (-1{\cdot}5)^3$

$$= 4{\cdot}640\,625.$$

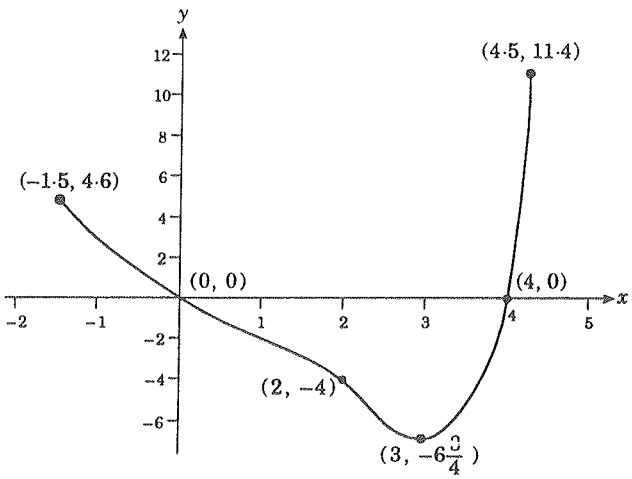
$x = 4{\cdot}5$, $y = \dfrac{1}{4}(4{\cdot}5)^4 - (4{\cdot}5)^3$

$$= 11{\cdot}390\,625.$$

Solving $y = \dfrac{1}{4}x^4 - x^3 = 0$

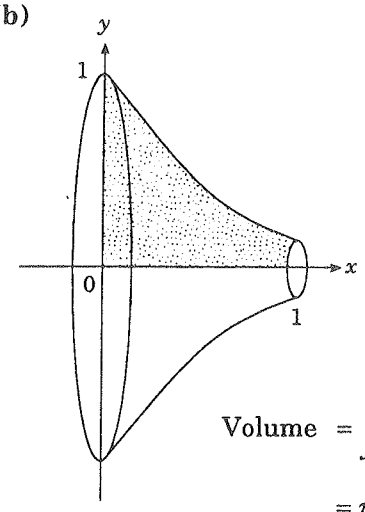
$$\therefore\quad x^3\left(\frac{x}{4} - 1\right) = 0$$

$\therefore\ x = 0,\ 4.$



(iv) Concave down for $0 < x < 2$.

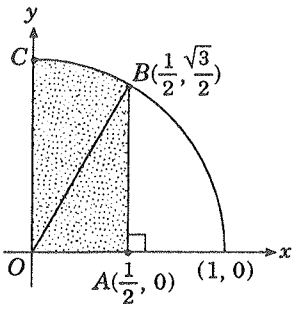
(b)



$$\text{Volume} = \int_0^1 \pi y^2\,dx$$
$$= \pi\int_0^1 (e^{-x})^2\,dx$$
$$= \pi\int_0^1 e^{-2x}\,dx$$
$$= -\frac{\pi}{2}\left[e^{-2x}\right]_0^1$$
$$= -\frac{\pi}{2}\left[e^{-2} - e^0\right]$$
$$= \frac{\pi}{2}\left(1 - e^{-2}\right).$$

QUESTION SEVEN

(a)



(i) $\angle COA = \dfrac{\pi}{2}$

$\angle BOA = \dfrac{\pi}{3}$ [sine of $B\hat{O}A$ is $\dfrac{1}{2}$ as $OB = 1$]

$\angle COB = \angle COA - \angle BOA$

$= \dfrac{\pi}{2} - \dfrac{\pi}{3}$

$= \dfrac{\pi}{6}$.

(ii) Area $\Delta OAB = \dfrac{1}{2} \times \dfrac{1}{2} \times \dfrac{\sqrt{3}}{2}$

$= \dfrac{\sqrt{3}}{8}$.

Area sector $COB = \dfrac{1}{2} r^2 \theta$

$= \dfrac{1}{2}(1)^2 \left(\dfrac{\pi}{6}\right)$

$= \dfrac{\pi}{12}$.

$\therefore$ Area $OABC = \dfrac{\pi}{12} + \dfrac{\sqrt{3}}{8}$

$= \dfrac{2\pi + 3\sqrt{3}}{24}$.

OR Area of sector of circle COB

$= \dfrac{30}{360} \times \pi r^2$, where radius $r = 1$,

$\therefore$ sector $COB = \dfrac{\pi}{12}$.

Area $\Delta OAB = \dfrac{1}{2} \times \dfrac{1}{2} \times \dfrac{\sqrt{3}}{2} = \dfrac{\sqrt{3}}{8}$

$\therefore$ Area $OABC = \dfrac{\sqrt{3}}{8} + \dfrac{\pi}{12}$

$= \dfrac{2\pi + 3\sqrt{3}}{24}$.

(b) (i) $y = \sqrt{1 - x^2}$

x	0	0·125	0·25	0·375	0·5
y	1	0·992	0·968	0·927	0·866

(ii) $\displaystyle\int_0^{\frac{1}{2}} \sqrt{1 - x^2}\, dx$

$= \dfrac{1}{3} \times 0{\cdot}125(1 + 4 \times 0{\cdot}992 + 2 \times 0{\cdot}968$

$+ 4 \times 0{\cdot}927 + 0{\cdot}866)$

$= 0{\cdot}478\,25$.

(iii) $\dfrac{2\pi + 3\sqrt{3}}{24} = 0{\cdot}478\,25$

$\therefore \quad 2\pi + 3\sqrt{3} = 24 \times 0{\cdot}478\,25$

$= 11{\cdot}478$

$2\pi = 6{\cdot}281847$

$\therefore \quad \pi = 3{\cdot}140\,92\cdots$

$\approx 3{\cdot}14$.

(c) (i) There are 20 different faces. Whatever face shows the first time, on the second roll there is a one in twenty chance that the same face shows.

$\therefore$ Probability of same letter $= \dfrac{1}{20}$.

(ii) On one roll, the probability of getting an E is $\dfrac{1}{20}$, and the probability of not getting an E is $\dfrac{19}{20}$.

We have E, E not E; E, not E, E; not E, E, E, and the probability is thus:

$$3 \times \frac{1}{20} \times \frac{1}{20} \times \frac{19}{20} = \frac{57}{8000}.$$

OR for 3 Unit students:

$$P(E) = \frac{1}{20}, \quad P(\tilde{E}) = \frac{19}{20}.$$

In 3 trials: $P(2 \text{ successes})$

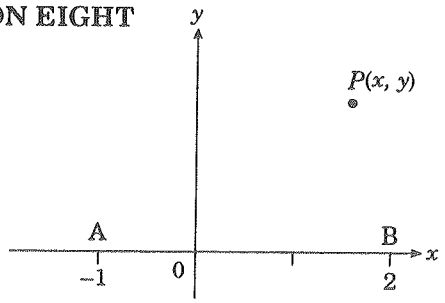
$= {}^3C_2 \left(\dfrac{1}{20}\right)^2 \left(\dfrac{19}{20}\right)^1$

$= 3 \times \dfrac{1}{20} \times \dfrac{1}{20} \times \dfrac{19}{20}$

$= \dfrac{57}{8000}$.

QUESTION EIGHT

(a)



(i) $PA^2 = (x+1)^2 + y^2$

$PB^2 = (x-2)^2 + y^2$.

(ii) $PA = 2PB$

$\therefore \quad PA^2 = 4PB^2$

$\therefore \quad (x+1)^2 + y^2 = 4\left[(x-2)^2 + y^2\right]$

$x^2 + 2x + 1 + y^2 = 4(x^2 - 4x + 4 + y^2)$

$= 4x^2 - 16x + 16 + 4y^2$

$\therefore \quad 3x^2 - 18x + 15 + 3y^2 = 0$

$x^2 - 6x + 5 + y^2 = 0$

$(x^2 - 6x + 9) + y^2 = 4$

$(x-3)^2 + y^2 = 2^2$

which is the equation of a circle.

(iii) Centre is (3, 0) and radius is 2.

(b) (i) At the end of the first year, the interest plus principal is \$50 000 (1·08).
He withdraws \$$M$, and thus has
$$\$50\,000 \times 1{\cdot}08 - M.$$

(ii) At the end of the 2nd year, after interest and withdrawal of \$$M$, he has
$$\$(50\,000 \times 1{\cdot}08 - M)\,1{\cdot}08 - M$$
$$= \$50\,000 \times 1{\cdot}08^2 - M(1 + 1{\cdot}08).$$
At the end of the 3rd year, he has
$$\$\left[50\,000 \times 1{\cdot}08^2 - M(1 + 1{\cdot}08)\right]1{\cdot}08 - M$$
$$= \$50\,000 \times 1{\cdot}08^3 - M(1 + 1{\cdot}08 + 1{\cdot}08^2).$$
At the end of the twentieth year, he has
$$\$50\,000 \times 1{\cdot}08^{20} - M(1 + 1{\cdot}08 + 1{\cdot}08^2 + \cdots + 1{\cdot}08^{19}).$$

(iii) The expression in brackets is a geometric series, and the sum is given by
$$S_{20} = \frac{a(r^n - 1)}{r - 1}, \quad \text{where } \begin{cases} n = 20 \\ r = 1{\cdot}08 \\ a = 1 \end{cases}$$
$$= \frac{1{\cdot}08^{20} - 1}{1{\cdot}08 - 1}$$
$$= \frac{1{\cdot}08^{20} - 1}{0{\cdot}08}.$$
Since he has \$0 at the end of the 20th year, we have:
$$50\,000 \times 1{\cdot}08^{20} - M \times \frac{(1{\cdot}08^{20} - 1)}{0{\cdot}08}$$
$$\therefore\ M = 50\,000 \times \frac{1{\cdot}08^{20} \times 0{\cdot}08}{(1{\cdot}08^{20} - 1)}$$
$$= 5092{\cdot}610\,441$$
$$\therefore\ M = 5092{\cdot}61.$$

(iv) If the interest was 10%:
$$M = 50\,000 \times \frac{1{\cdot}1^{20} \times 0{\cdot}1}{(1{\cdot}1^{20} - 1)}$$
$$= \$5872{\cdot}98.$$

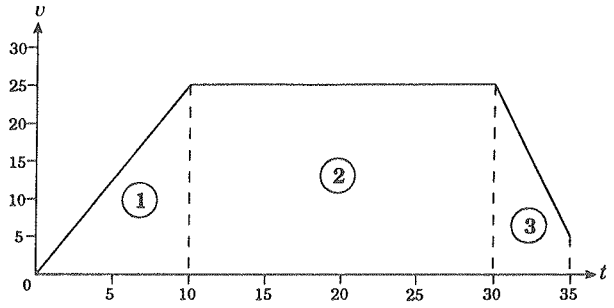
If the interest was 15%:
$$M = 50\,000 \times \frac{1{\cdot}15^{20} \times 0{\cdot}15}{(1{\cdot}15^{20} - 1)}$$
$$= \$7988{\cdot}07 \text{ (approximately \$8000)}.$$

If the interest was 16%:
$$M = 50\,000 \times \frac{1{\cdot}16^{20} \times 0{\cdot}16}{(1{\cdot}16^{20} - 1)}$$
$$= \$8433{\cdot}35.$$

Thus, for M to be \$8000, the interest needs to be above 15%, but closer to 15% than to 16%.

QUESTION NINE

(a) (i)



(ii) ① For $0 \le t \le 10$, $v = 2{\cdot}5t$
$$\therefore\ s = \frac{2{\cdot}5t^2}{2} + c.$$
When $t = 0$, $s = 0$, $\therefore\ c = 0$
$$\therefore\ s = 1{\cdot}25t^2.$$
When $t = 10$, $s = 125$.

② For $10 \le t \le 30$, distance covered is 20×25 m $= 500$ m.
$\therefore$ when $t = 30$, $s = 125 + 500$
$= 625$.

③ For $30 \le t \le 35$, $v = -4t + c$.
When $t = 30$,
$$25 = -4 \times 30 + c$$
$$\therefore\ c = 145$$
$$\therefore\ v = -4t + 145$$
$$\therefore\ s = \frac{-4t^2}{2} + 145t + K.$$
$$= -2t^2 + 145t + K.$$
When $t = 30$, $s = 625$
$$\therefore\ 625 = -2 \times 900 + 145 \times 30 + K$$
$$\therefore\ K = -1925$$
$$\therefore\ s = -2t^2 + 145t - 1925,$$
$\therefore$ when $t = 35$,
$$s = -2 \times 35^2 + 145 \times 35 - 1925$$
$$= 700.$$

OR $t = 0$, $s = 0$
For $t = 10$, $s =$ Area ①
$$= \frac{1}{2} \times 10 \times 25$$
$$= 125.$$
For $0 \le t \le 10$, speed is increasing,
so $\dfrac{d^2s}{dt^2} > 0$, curve is concave up.
For $t = 30$, $s =$ Area ① + Area ②
$$= 125 + 25 \times 20$$
$$= 625.$$
For $10 \le t \le 30$, speed is constant, so s in terms of t is a straight line.
For $t = 35$,
$s =$ Area ① + Area ② + Area ③

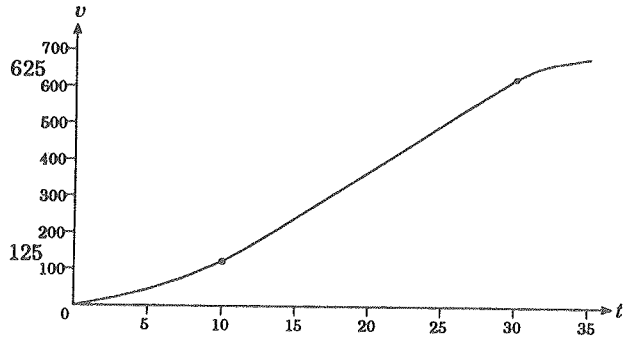
$$= 625 + \frac{1}{2} \times 5 \times 20 + 5 \times 5$$

(↑ triangle, ↑ rectangle)

$$= 700.$$

For $30 \le t \le 35$, speed is decreasing,

$\frac{d^2s}{dt^2} < 0$, concave down.



(b) $y = A\sin(at+b)$

(i) $A = 1{\cdot}73$.

(ii) When $t = 0$, $y = A\sin b$

$$\therefore \quad 1{\cdot}35 = 1{\cdot}73\sin b$$

$$\sin b = \frac{1{\cdot}35}{1{\cdot}73}$$

$$\therefore \quad b = 0{\cdot}895\,22\cdots$$

(iii) Period $T = \frac{2\pi}{n}$

$$\therefore \quad n = \frac{2\pi}{T}$$

$$= \frac{2\pi}{12{\cdot}42}$$

$$= 0{\cdot}505\,892\cdots$$

(c) $2\ln x = \ln(5+4x)$

$\therefore \ln x^2 = \ln(5+4x)$

$\therefore \quad x^2 = 5+4x$

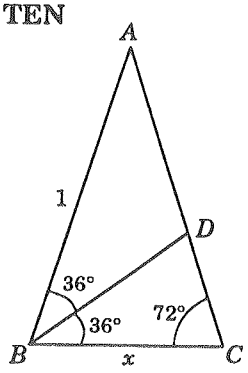
i.e. $x^2 - 4x - 5 = 0$

$(x-5)(x+1) = 0$

$\therefore \ x = 5, \ -1.$

QUESTION TEN

(a) (i)



(ii) In ΔABC, $\angle ABC = \angle ACB = 72°$, and $\angle BAC = 36°$ (angle sum of a triangle).
In ΔBCD, $\angle BDC = 72°$ (angle sum of a triangle),

$\therefore \ \angle BDC = \angle BCD = 72°$

and $\angle DBC = 36°$,

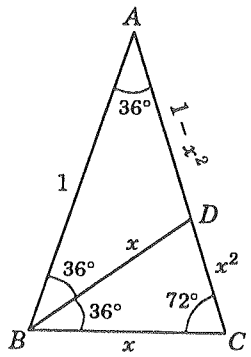
$\therefore \ \Delta ABC \| \Delta BCD$ (equiangular).

(iii) Since the two triangles are equiangular, the corresponding sides are in the same ratio.

$$BC : AB = x : 1$$

$$\therefore \quad CD : BC = x : 1.$$

Since $BC = x$, we have $CD = x^2$.
Now, since $AC = 1$, we have $AD = 1 - x^2$.



In ΔABD, $\angle ABD = \angle BAD = 36°$,

$$\therefore \quad AD = BD$$

$$1 - x^2 = x$$

$$x^2 + x - 1 = 0$$

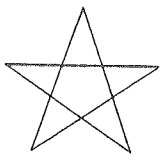
$$\therefore \quad x = \frac{-1 \pm \sqrt{1+4}}{2}$$

$$= \frac{-1 \pm \sqrt{5}}{2}.$$

Since $x > 0$, we have $x = \frac{\sqrt{5}-1}{2}$.

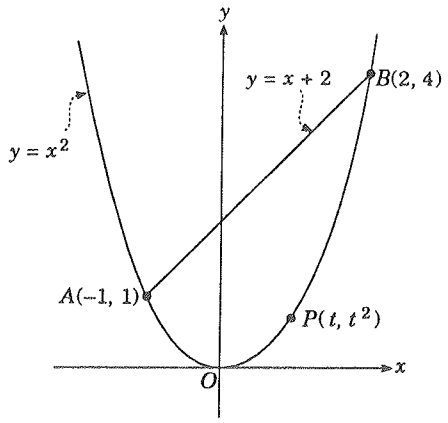
A NOTE OF INTEREST:

$x : 1$ is known as the Golden Ratio. It is used in the construction of a regular pentagon and is everywhere present in the pentagram.



The golden rectangle (used for its natural beauty in art and architecture) has its length : width ratio that of the Golden Ratio.

(b)



(i) Area of parabolic segment

$$= \int_{-1}^{2} (x+2-x^2)\,dx$$

$$= \left[\frac{x^2}{2}+2x-\frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{4}{2}+4-\frac{8}{3}-\left(\frac{1}{2}-2+\frac{1}{3}\right)$$

$$= 4\frac{1}{2} \text{ units}^2.$$

(ii) To find the area of ΔAPB, find the length of AB, then the perpendicular distance of P from the line AB.

Length of $AB = \sqrt{(2+1)^2+(4-1)^2}$

$$= \sqrt{18}$$

$$= 3\sqrt{2}.$$

Length of line from P perpendicular to AB: $(x-y+2=0)$

$$d = \frac{t-t^2+2}{\sqrt{1^2+1^2}}$$

$$= \frac{t-t^2+2}{\sqrt{2}}.$$

Area of $\Delta ABP = \frac{1}{2} \times AB \times d$

$$= \frac{1}{2} \times 3\sqrt{2} \times \frac{(t-t^2+2)}{\sqrt{2}}$$

$\therefore \quad A = \frac{3}{2}(t-t^2+2).$

To find the maximum area, find $\frac{dA}{dt}$.

$$\frac{dA}{dt} = \frac{3}{2}(1-2t)$$

$$= 0 \text{ when } t = \frac{1}{2}.$$

$$\frac{d^2A}{dt^2} = \frac{3}{2} \times -2$$

$$= -3.$$

Since this is negative, area is a maximum value when $t = \frac{1}{2}$,

$\therefore P$ is the point $\left(\frac{1}{2}, \frac{1}{4}\right)$.

$$\therefore \text{ Area} = \frac{3}{2}\left(\frac{1}{2}-\frac{1}{4}+2\right)$$

$$= \frac{27}{8}$$

$$= 3\frac{3}{8} \text{ units}^2.$$

Now $\frac{3}{4}$ of area of segment $= \frac{3}{4} \times 4\frac{1}{2}$

$$= 3\frac{3}{8}.$$

i.e. Result is shown.

COMMENTS ON THE 1993 2/3 UNIT PAPER

2. (b) This is a standard technique for estimating the size of fish populations. Similar examples could be discussed when teaching proportions or probability. It underlines the notion of probability as 'relative frequency'. Perhaps the number of a particular colour of marbles in a bag could be estimated by this process in the classroom.

3. (b) Examples of this type occur regularly in everyday life. Other variables which might be considered include the number of unemployed people in Australia; the number of megalitres of water consumed annually in Sydney; the number of school students with influenza; a person's temperature when sick and under medication; and the amount of plastics produced in Australia. Students

could either be given the graph of the resultant function or be asked to produce such a graph.

Similar questions could be asked by giving the graph of the resultant function and requiring students to write an explanation from it, or, by giving an explanation, after which students must reproduce a graph.

5. (b) Similar types of graphs are easily devised which either exploit a result on the table of integrals or a known area formula. If the regions between the curve and the x axis are shaded, the class could be asked to work out the total area of the shaded regions as well as the value of the integral over the interval. Students could be simultaneously asked to note where f' is positive, zero, or negative, and invited to draw in tangents at those points where f' exists.

8. (b) (iv) Although this question implies that students find the value of r by trial and error on their calculators, 3 Unit and 4 Unit students could extend the question.

If P = principal invested, $100r$ = annual percentage rate, M = annuity to be paid, and n = number of years of annuity, the general problem is to solve:

$$Pr(1+r)^n - M\Big((1+r)^n - 1\Big) = 0$$

in terms of r, given P, M and n. Three-unit students might try Newton. Four-unit students do better to observe that:

$$r = \frac{M}{P}\Big(1-(1+r)^{-n}\Big),$$

and thus try the iterative procedure:

$$r_{k+1} = \frac{M}{P}\Big(1-(1+r_k)^{-n}\Big).$$

If $\dfrac{M}{P} = 0{\cdot}16$ and $n = 20$, then, letting

$$r_1 = 0{\cdot}1, \qquad r_2 = 0{\cdot}1362,$$
$$r_3 = 0{\cdot}1476, \quad r_4 = 0{\cdot}1498, \cdots$$

But why does this work?

Students should graph

$$y = r \text{ and } y = \frac{M}{P}\Big(1-(1+r)^{-n}\Big),$$

and then remember they are searching for the point of intersection.

9. (a) Conversely, a class could be given a sketch of the speed $v = v(t)$ and then asked to describe the motion in words. From the given graph of v, the class could be asked to sketch the displacement $s = s(t)$ and acceleration $a = a(t)$.

(b) Oceanographers call this effect M_2. The similar tidal effect due to the Sun is called S_2. The tidal charts published in newspapers essentially show $M_2 + S_2$ and, unfortunately, can be well approximated by sine curves only at certain dates and for short periods of time. The example shows that naturally occurring periodic phenomena can often be approximated by sine curves.

(c) The largest number which is currently known to be prime is $2^{756\,839} - 1$. A serious and rather beautiful application of logarithms is evaluation of the number of digits in the decimal expansion of such a number. Students should begin by evaluating $\log_{10} 2^{756\,839}$.

10. (a) The diagram must have been considered by the Greeks, though it doesn't seem to be in Euclid. It is natural to continue to find the exact value of $\cos 36°$. Knowledge of this value reveals that the regular pentagon can be constructed by ruler and compass. Good 3 Unit students might enjoy doing such a construction. [Remember the result of Gauss (1796), proved when he was barely eighteen, that if p is prime, the regular p-gon is constructible by ruler and compass, if and only if:

$$p = 2^{2^n} + 1 \text{ for } n = 0,\ 1,\ 2,\ 3,\ \ldots].$$

The value of $\cos 36°$ is also half the Golden Ratio, so there is a natural opportunity to discuss aesthetics and mystery in mathematics.

(b) This is a specific instance of the general result of Archimedes (circa 250 BC) that the area of a parabolic segment is always four-thirds times the maximum area of the inscribed triangle. This geometric theorem is equivalent to the modern statement:

$$\int_0^a x^2\ dx = \frac{a^3}{3}.$$

[What was the first curvilinear area for which an exact expression for the area was obtained? How many centuries after Archimedes' work was it proved that:

$$\int_0^a x^3\ dx = \frac{1}{4}a^4\ ?]$$

□