1992 HIGHER SCHOOL CERTIFICATE SOLUTIONS

2/3 UNIT MATHEMATICS

QUESTION ONE

(a) $\text{Average} = \dfrac{75+29+80+42+65+59+38}{7}$

$= \dfrac{388}{7}$

$= 55{\cdot}4$ (one decimal place).

(b) $x^2+5x-6 = (x+6)(x-1)$

(c) $4x+7 = 3(x-2)$

$= 3x-6$

$x+7 = -6$

$x = -6-7$

$x = -13$

(d) $\angle EDC = \angle BAD$ (alternate angles $AB \parallel CD$)

$= 60°$

$\therefore \quad \angle ECD = \theta$

$= 180-70-60$ (angle sum of ΔECD is 180°)

$= 50°.$

(e) $x+y = 2$

$2x-y = 7$

$\therefore \quad 3x = 9$

$\therefore \quad x = 3$

Sub $x = 3$ in $x+y = 2$

$\therefore \quad y = -1$

$\therefore$ Solution is $x = 3,\ y = -1.$

(f) $\text{Amount} = \$10\ 000\left(1+\dfrac{12}{100}\right)^6$

$= \$10000(1{\cdot}12)^6$

$= \$19738{\cdot}23$ (nearest cent).

QUESTION TWO

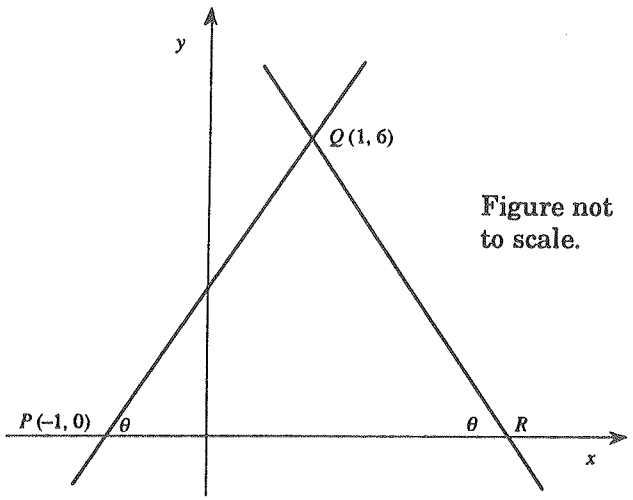


Figure not to scale.

(a) Mid-point of $PQ = \left(\dfrac{x_1+x_2}{2},\ \dfrac{y_1+y_2}{2}\right)$

$= \left(\dfrac{-1+1}{2},\ \dfrac{0+6}{2}\right)$

$= (0,\ 3).$

(b) Equation of PQ:

$y-y_1 = \dfrac{y_2+y_1}{x_2+x_1}(x-x_1)$

$y-6 = \dfrac{0-6}{-1-1}(x-1)$

$y-6 = \dfrac{-6}{-2}(x-1)$

$= 3(x-1)$

$= 3x-3$

$\therefore \quad y = 3x+3$ as required.

(c) θ is the angle PQ makes with positive x-axis.

$\therefore \quad \tan\theta = \text{gradient } PQ$

$= \dfrac{6-0}{1--1}$

$$= \frac{6}{2}$$

$\therefore \quad \tan\theta = 3$ as required.

(d) Gradient $QR = \tan(180° - \theta)$

$$= -\tan\theta$$

$$= -3.$$

(e) Equation of QR is

$$y - y_1 = m(x - x_1)$$

i.e. $\quad y - 6 = -3(x - 1)$

i.e. $\quad y - 6 = -3x + 3$

i.e. $\quad 3x + y - 9 = 0$

(f) At R, $y = 0$

$\therefore \quad 3x + 0 - 9 = 0$

$\therefore \quad 3x = 9$

$\therefore \quad x = 3$

$\therefore$ R is $(3, 0)$.

(g) P is $(-1, 0)$, QR is $3x + y - 9 = 0$

$\therefore$ Perpendicular distance from P to QR

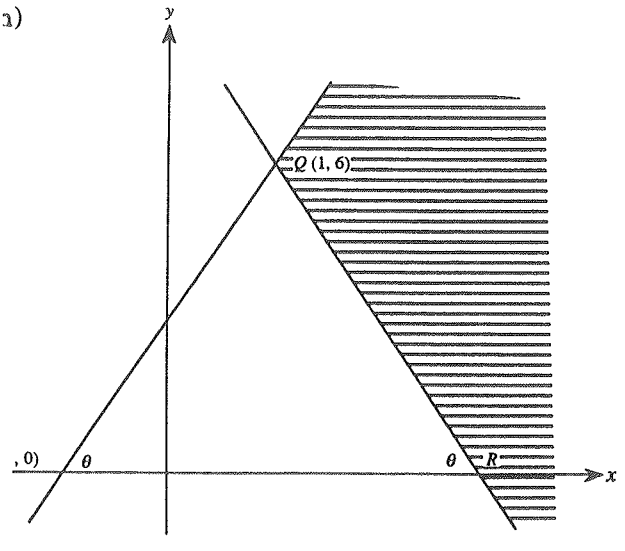
$$= \left|\frac{3x_1 + y_1 - 9}{\sqrt{3^2 + 1^2}}\right|$$

$$= \left|\frac{3x(-1) + 0 - 9}{\sqrt{10}}\right|$$

$$= \left|\frac{-3 - 9}{\sqrt{10}}\right|$$

$$= \frac{12}{\sqrt{10}}.$$

a)



QUESTION THREE

(a)(i) $\quad \frac{d}{dx}e^{3x} = 3e^{3x}$

(ii) $\quad \frac{d}{dx}\ln(5x - 1) = \frac{5}{5x - 1}$

(iii) $\quad \frac{d}{dx}x\tan x = \tan x . 1 + x . \sec^2 x$

$$= \tan x + x\sec^2 x$$

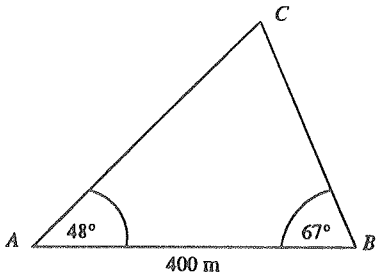
(b) (i) $\quad \int (2x + 3)^{10}\,dx = \frac{(2x + 3)^{11}}{2 \times 11} + \text{constant}$

$$= \frac{(2x + 3)^{11}}{22} + \text{constant}$$

(ii) $\quad \int \sin\frac{x}{2}\,dx = \frac{-\cos\frac{x}{2}}{\frac{1}{2}} + \text{constant}$

$$= -2\cos\frac{x}{2} + \text{constant}$$

(c) $LC = 65°$ (angle sum of ΔABC)



By Sine Rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

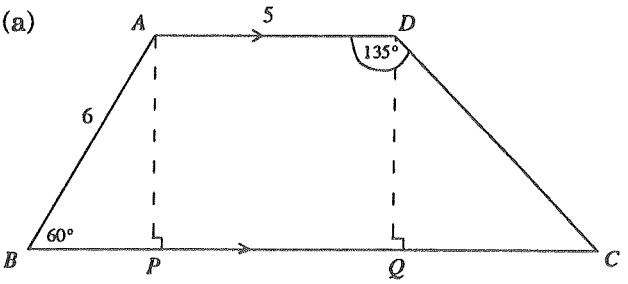
i.e. $\quad \frac{a}{\sin 48°} = \frac{400}{\sin 65°}$

$\therefore \quad a = \frac{400\sin 48°}{\sin 65°}$

$\therefore \quad BC = 328$ m (nearest metre)

QUESTION FOUR

(a)



(i) In ΔABP, $\dfrac{BP}{AB} = \cos 60°$

$BP = AB.\cos 60°$

$= 6 \times \dfrac{1}{2}$

$\therefore \quad BP = 3$ units as required

(ii) In ΔABP, $\dfrac{AP}{AB} = \sin 60°$

$\therefore \quad AP = 6 \sin 60°$

$= 6 \times \dfrac{\sqrt{3}}{2}$

$= 3\sqrt{3}$

(iii) $BC = BP + PQ + QC$ (addition of intervals)

Since $\angle ADC = 135°$

$= \angle ADQ + \angle QDC$

$= 90° + \angle QDC$ ($\angle ADQ$, $\angle DQC$ alternate $\angle$s, $AD \parallel BC$)

$\therefore \quad \angle QDC = 45°$.

Thus ΔQDC is right angled and isosceles.

Hence $QD = QC = AP$.

So $BC = BP + PQ + QC$

$= 3 + AD + 3\sqrt{3}$ ($AD = PQ$, $ADPQ$ is a rectangle)

$= 3 + 5 + 3\sqrt{3}$

$= 8 + 3\sqrt{3}$

(iv) The area of a trapezium is $1/2 \times$ sum of parallel sides $\times$ perpendicular distance between them.

$\therefore$ area $ABCD = \dfrac{1}{2} \times \left(5 + 8 + 3\sqrt{3}\right) \times 3\sqrt{3}$

$= \dfrac{1}{2} \times \left(13 + 3\sqrt{3}\right) \times 3\sqrt{3}$

$= \dfrac{1}{2} \times \left(39\sqrt{3} + 27\right)$ unit^2

(b) (i) Probability

$= P[(1, 1)$ or $(2, 2)$ or $(3, 3)$ or $(4, 4)$ or $(5, 5)$ or $(6, 6)]$

$= \dfrac{1}{36} + \dfrac{1}{36} + \dfrac{1}{36} + \dfrac{1}{36} + \dfrac{1}{36} + \dfrac{1}{36}$

$= \dfrac{6}{36}$

$= \dfrac{1}{6}$.

(ii) Let Pat be first, Chris be second in tossing. Sample space:

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

P (Chris' no. > Pat's no.) $= \dfrac{15}{36} = \dfrac{5}{12}$

(c) $A = 10e^{-kt}$

(i) $t = 4$, $A = 3{\cdot}2$

$\therefore \quad 3{\cdot}2 = 10e^{-k \times 4}$

$\dfrac{3{\cdot}2}{10} = e^{-4k}$

$\ln 0{\cdot}32 = -4k$

$k = \dfrac{\ln 0{\cdot}32}{-4}$

$= 0{\cdot}28$ (two decimal places)

(ii) $A = 1$

$\therefore \quad 1 = 10e^{-kt}$

$0{\cdot}1 = e^{-kt}$

$\ln 0{\cdot}1 = -kt$

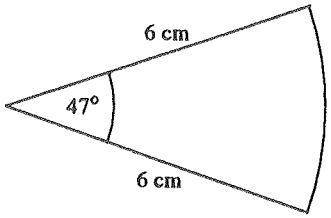
$\therefore \quad t = \dfrac{\ln 0{\cdot}1}{-k}$

$= \dfrac{\ln 0{\cdot}1}{-0{\cdot}28}$

$= 8$ hours (nearest hour)

QUESTION FIVE

(a) $A = \dfrac{1}{2} r^2 \theta$ where θ is in radians.



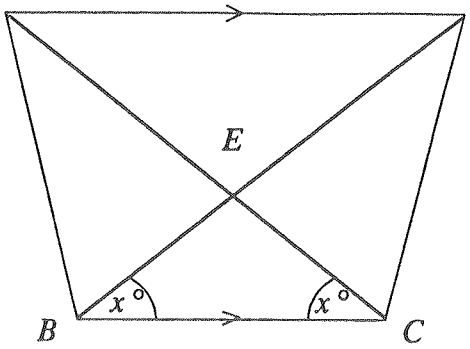
$180° = \pi$ rads.

$\therefore \quad 47° = \dfrac{\pi}{180} \times 47$ rads.

$\therefore \quad A = \dfrac{1}{2} \times 6^2 \times \dfrac{\pi}{180} \times 47$

$\therefore$ area $= 15$ cm^2 (nearest square cm)

(b)



(i) $\angle ADE = \angle CBE$ ($AD \parallel BC$, alternate $\angle$s)
$= x°$

Similarly, $\angle DAE = x°$

$\therefore$ $AE = DE$ (sides opposite equal $\angle$s ADE, DAE)

(ii) In ΔBEC, $EB = EC$ (opposite equal $\angle$s EBC, ECB)

But $AC = AE + EC$

and $DB = DE + EB$

$\therefore$ $AC = DB$

In Δs ABC, DCB

$AC = DB$ (as above)

BC is common

$\angle ACB = \angle DBC$ (given)

$\therefore$ $\Delta ABC \equiv \Delta DCB$ (SAS)

(iii) $\therefore$ $\angle ABC = \angle DCB$ (corresponding $\angle$s in congruent Δs)

i.e. $\angle ABD + \angle DBC = \angle DCA + \angle ACB$

But $\angle DBC = \angle ACB$ (given)

$\therefore$ $\angle ABD = \angle DCA$.

(c) 1st layer (top) has $5 + 0$ logs
2nd layer has $5 + 1$ logs
3rd layer has $5 + 2$ logs

(i) nth layer (bottom) has $5 + (n - 1)$ logs,
i.e. $4 + n$ logs

(ii) AP $a = 5$, $l = 4 + n$ no. of terms is n

$$S_n = \frac{n}{2}[a + l]$$

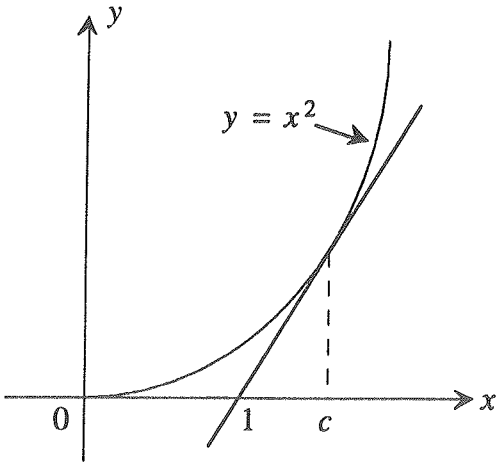
$$= \frac{n}{2}[5 + 4 + n]$$

$$= \frac{n}{2}[9 + n]$$

i.e. There are $\frac{1}{2}n(n+9)$ logs in the stack.

QUESTION SIX

(a)



(i) $y = x^2$

$\frac{dy}{dx} = 2x$

$\therefore$ grad. of tangent at $x = c$ is given by $2c$.

(ii) At $x = c$, $y = c^2$

$\therefore$ equation of tangent at $x = c$ is given by $y - c^2 = 2c(x - c)$,

i.e. $y = 2cx - c^2$

(iii) $(1, 0)$ lies on tangent $y = 2cx - c^2$

$$0 = 2c \times 1 - c^2$$

i.e. $c^2 - 2c = 0$

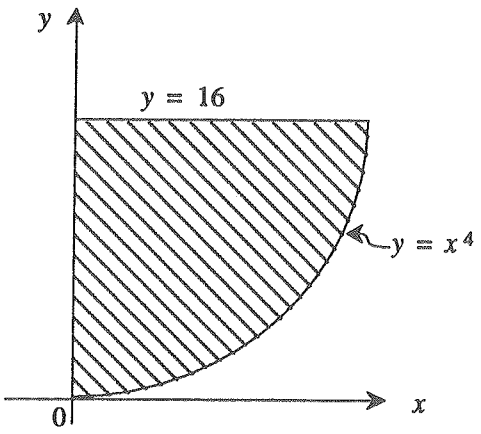
i.e. $c(c - 2) = 0$

$\therefore$ $c = 0$ or $c = 2$

$c = 0$ is a trivial solution

$\therefore$ solution is $c = 2$.

(b)



$$V = \pi \int_0^{16} x^2 \, dy$$

$$= \pi \int_0^{16} y^{\frac{1}{2}} \, dy \qquad \left(y = x^4 \text{ so } y^{\frac{1}{2}} = x^2 \right)$$

$$V = \pi\left[\frac{2y^{\frac{3}{2}}}{3}\right]_0^{16}$$

$$= \pi\left[\frac{2\times 16^{\frac{3}{2}}}{3} - \frac{2\times 0^{\frac{3}{2}}}{3}\right]$$

$$= \pi \times \frac{128}{3}.$$

$\therefore$ Volume $= \frac{128}{3}\pi$ unit^3.

(c) (i) $v = \frac{dx}{dt}$

$\therefore \quad x = \int_a^b v\,dt$

Since the initial time is 0, $a = 0$.
The unit for velocity was km/h.
The final time was 4 mins. or 1/15 h.

$\therefore \qquad b = \frac{1}{15}$

$\therefore \qquad x = \int_0^{\frac{1}{15}} v\,dt.$

(ii) Using the formula:

$$\int_0^{\frac{1}{15}} f(t)\,dt$$

$$\approx \frac{h}{3}\left\{y_0 + y_4 + 4\times(y_1 + y_3) + 2\times y_2\right\}$$

where $h = \frac{1}{4}\left(\frac{1}{15} - 0\right)$ i.e. $\frac{1}{60}$

$$x \approx \frac{1}{3}\times\frac{1}{60}\{0 + 40 + 4\times(25 + 30) + 2\times 34\}$$

$$\approx \frac{1}{180}\{328\}$$

$\approx$ 1·8 km (one decimal place)

QUESTION SEVEN

(a) $3^{2x} + 2.3^x - 15 = 0$

Let $u = 3^x$, $u^2 = 3^{2x}$

Then $u^2 + 2u - 15 = 0$

i.e. $(u + 5)(u - 3) = 0$

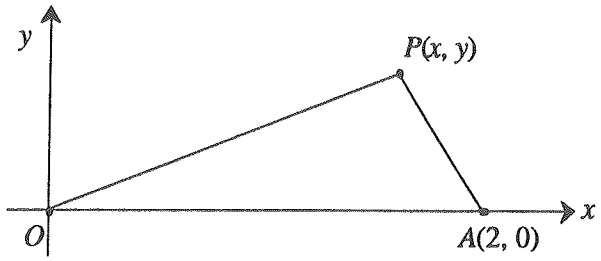
$\therefore \quad u = -5$ or $u = 3$

$\therefore \quad 3^x = -5$ or $3^x = 3$

if $3^x = -5$, no solution

if $3^x = 3$, $x = 1$

(b)



(i) Grad. $AP = \frac{y-0}{x-2} = \frac{y}{x-2}$

(ii) Grad. $OP = \frac{y}{x}$

But $AP \perp OP$ $\therefore$ product of grads. is -1

i.e. $\frac{y}{(x-2)}\times\frac{y}{x} = -1$

i.e. $y^2 = -x(x-2)$

i.e. $x^2 - 2x + y^2 = 0$

(iii) i.e. $x^2 - 2x + 1 + y^2 = 1$

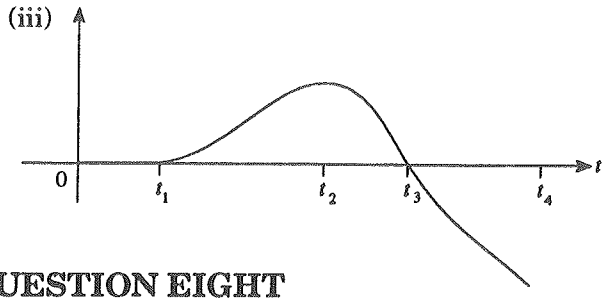
i.e. $(x-1)^2 + y^2 = 1$

This is the equation of a circle centre (1, 0) radius 1 unit.

(c) (i) Moving faster at t_2 since the gradient of the tangent at t_2 is greater than the gradient of the tangent at t_1 and the gradient of the tangent at any point is $\frac{dx}{dt}$ (or the velocity) at that point.

(ii) At $t = 0$, dx/dt appears to be zero since the tangent at $t = 0$ appears to be horizontal.
$\therefore$ Velocity at $t = 0$ appears to be zero.

(iii)



QUESTION EIGHT

(a) $P = -t^3 + 6t^2 + 25 \qquad 0 \le t \le 6$

$\frac{dP}{dt} = -3t^2 + 12t$

$\frac{d^2P}{dt^2} = -6t + 12$

(i) At $\frac{dP}{dt} = 0$, $\quad -3t(t-4) = 0$

$\therefore \quad t = 0$ or $t = 4$

At $t = 0$, $\quad \frac{d^2P}{dt^2} > 0$

At $t = 4$, $\quad \frac{d^2P}{dt^2} < 0$

$\therefore$ Maximum P at 4 years.

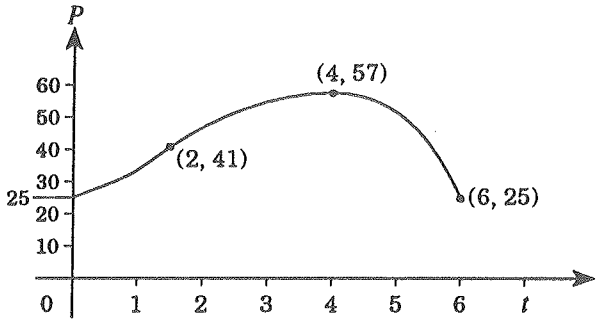
(ii) Maximum P is P at $t = 4$

$\therefore \quad P = -4^3 + 6 \times 4^2 + 25$

$= 57$

$\therefore$ Maximum P is 57 kangaroos.

(iii)



(iv) Population increasing most rapidly when dP/dt was greatest

i.e. $\quad \frac{d^2P}{dt^2} = 0$

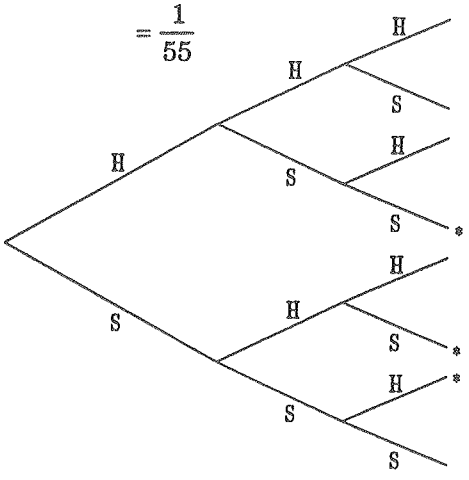
$-6t + 12 = 0$

This occurs at $t = 2$ years.

(b) (i) $P(\text{1st hard chocolate}) = \frac{4}{12} = \frac{1}{3}$

(ii) $P(\text{HHH}) = \frac{4}{12} \times \frac{3}{11} \times \frac{2}{10}$

$= \frac{1}{55}$



(iii) $P(\text{exactly one hard chocolate})$

$= \frac{4}{12} \times \frac{8}{11} \times \frac{7}{10} + \frac{8}{12} \times \frac{4}{11} \times \frac{7}{10} + \frac{8}{12} \times \frac{7}{11} \times \frac{4}{10}$

$= \frac{28}{55}$

QUESTION NINE

(a) (i) To find A:

$\sin x = \cos x$

i.e. $\quad \frac{\sin x}{\cos x} = 1$

i.e. $\quad \tan x = 1$

$\therefore \quad x = \frac{\pi}{4}$

$\therefore$ coordinates of A: $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$

To find B:

$\sin x = 0$

$\therefore \quad x = \pi$

$\therefore \quad B$ is $(\pi, 0)$

(ii) Shaded area under x-axis is same area as unshaded area above x-axis under curve $y = \cos x$.

$\therefore$ Area $= \int_0^{\frac{\pi}{4}} \cos x \, dx + \int_{\frac{\pi}{4}}^{\pi} \sin x \, dx$

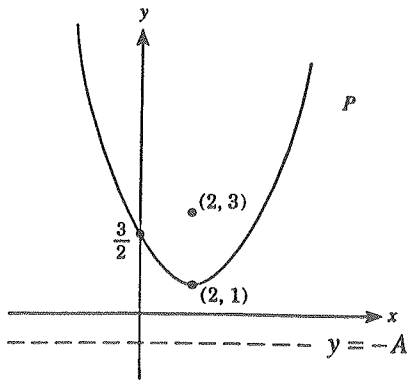
$= \left[\sin x\right]_0^{\frac{\pi}{4}} + \left[-\cos x\right]_{\frac{\pi}{4}}^{\pi}$

$= \sin\frac{\pi}{4} - \sin 0 - \left[\cos\pi - \cos\frac{\pi}{4}\right]$

$= \frac{1}{\sqrt{2}} - 0 - \left[-1 - \frac{1}{\sqrt{2}}\right]$

$= 1 + \frac{2}{\sqrt{2}} = 1 + \sqrt{2}$ units.

(b) (i)



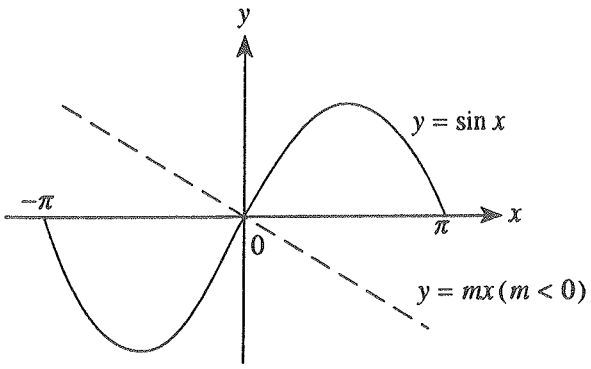
(ii) Vertex is (2, 1).
Focal length is 2.
Equation is:

$$(x-h)^2 = 4a(y-k)$$

i.e. $(x-2)^2 = 4\times 2\times(y-1)$

i.e. $(x-2)^2 = 8(y-1)$

(c)



From the diagram, $\sin x = mx = 0$ at $x = 0$

Now $\sin x \le 0$ for $-\pi \le x \le 0$

and $mx > 0$ for $-\pi \le x < 0$, $m < 0$

Likewise $\sin x \ge 0$ for $0 \le x \le \pi$

and $mx < 0$ for $0 < x \le \pi$, $m < 0$

So $x = 0$ is the only solution satisfying $-\pi \le x \le \pi$.

QUESTION TEN

(a) (i) The series $a + ar + ar^2 + \cdots$ has a limiting sum if $-1 < r < 1$.

The limiting sum S is given by $S = \dfrac{a}{(1-r)}$.

(ii) If $r = \dfrac{1}{w}$ and $S = \dfrac{1}{(1-w)}$

$$\frac{1}{(1-w)} = \frac{a}{\left(1-\frac{1}{w}\right)}$$

i.e. $\dfrac{1}{(1-w)} = \dfrac{aw}{(w-1)}$

i.e. $\dfrac{1}{(1-w)} = \dfrac{-aw}{(1-w)}$

$\therefore \quad 1 = -aw$

$\therefore \quad a = \dfrac{-1}{w}$.

$\therefore$ Series is $-\dfrac{1}{w} - \dfrac{1}{w^2} - \dfrac{1}{w^3} - \dfrac{1}{w^4} - \cdots$

(b) (i) Using speed $= \dfrac{\text{distance}}{\text{time}}$

$\therefore \quad \text{time} = \dfrac{\text{distance}}{\text{time}}$

$\therefore \quad \text{time} = \dfrac{1000}{v}$.

But cost per hour

= driver cost/h + fuel cost/h

$$= 2\times 20 + \left(6 + \frac{v^2}{50}\right)\times 0{\cdot}50$$

$$= 40 + 3 + \frac{v^2}{100}$$

$$= 43 + \frac{v^2}{100}.$$

$\therefore$ Total cost = cost per hour × no. of hours.

$$C = \left(43 + \frac{v^2}{100}\right)\times\frac{1000}{v}$$

$\therefore \quad C = \dfrac{43\,000}{v} + 10v$

(ii) $\dfrac{dC}{dv} = \dfrac{-43\,000}{v^2} + 10$

$\dfrac{dC}{dv} = 0, \dfrac{-43\,000}{v^2} + 10 = 0$

$\therefore \quad 10v^2 = 43\,000$

$\therefore \quad v^2 = 4300$

$\therefore \quad v = \sqrt{4300}$ (ignore negative v)

$= 65{\cdot}57.$

The time for the trip is to be less than 12 hours.

i.e. $\dfrac{1000}{v} \le 12$

$$v \ge \frac{1000}{12} \ge 83\frac{1}{3}.$$

This means that the solution found from $\dfrac{dC}{dv} = 0$ does not satisfy the time restraint.

Also, we are given $v \le 100$.

Calculating cost for $v = 83\frac{1}{3}$ and $v = 100$,

when $v = 100$, $C = 10\times 100 + \dfrac{43\,000}{100}$

$= \$1430,$

when $v = 83\frac{1}{3}$, $C = 10\times 83\frac{1}{3} + \dfrac{43\,000}{83\frac{1}{3}}$

$= \$1349{\cdot}33.$

$\therefore$ Speed to minimise cost is $83\frac{1}{3}$ km/h.