1991 HIGHER SCHOOL CERTIFICATE SOLUTIONS

2/3 UNIT MATHEMATICS

QUESTION ONE

(a) $0{\cdot}362\,34\ldots = 0{\cdot}4$ (to one decimal place)

(b) $(3x)^2 - y^2 = (3x - y)(3x + y)$

(c) $\dfrac{1}{3-\sqrt{2}} \times \dfrac{3+\sqrt{2}}{3+\sqrt{2}} = \dfrac{3+\sqrt{2}}{(3-\sqrt{2})(3+\sqrt{2})}$

$= \dfrac{3+\sqrt{2}}{3^2 - (\sqrt{2})^2}$

$= \dfrac{3+\sqrt{2}}{9-2} = \dfrac{3+\sqrt{2}}{7}$

(d) $60 = \dfrac{a}{1-1/2}$

$\therefore \dfrac{a}{1/2} = 60$

$a = 60 \times \dfrac{1}{2}$

$a = 30$

(e) $\dfrac{2x}{3} - \dfrac{x}{5} = 1$ $\qquad$ $\dfrac{2x}{3} - \dfrac{x}{5} = 1$

Either $\qquad$ or

$\dfrac{10x}{15} - \dfrac{3x}{15} = 1$ $\qquad$ $\dfrac{15 \times 2x}{3} - \dfrac{15x}{5} = 1 \times 15$

$\dfrac{7x}{15} = 1$ $\qquad$ $10x - 3x = 15$

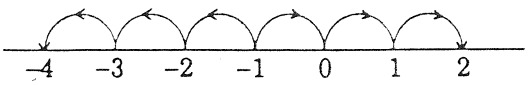
$7x = 15$ $\qquad$ $7x = 15$

$x = \dfrac{15}{7}$ $\qquad$ $x = 2\dfrac{1}{7}$ or $\dfrac{15}{7}$

$= 2\dfrac{1}{7}$

(f) $|x+1| \le 3$ $\qquad$ $|x-(-1)| \le 3$

This means that x is such that, on a number line, the distance between 'x' and '-1' is less than or equal to 3 units, i.e.



i.e. between -4 and 2



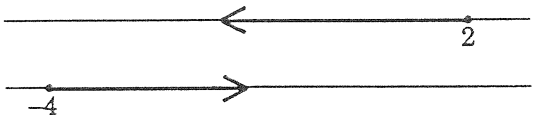
OR

$|x+1| \le 3$

i.e. $x + 1 \le 3$ or $-(x+1) \le 3$

$x + 1 \le 3$ or $x + 1 \ge -3$

$x \le 2$ or $x \ge -4$



i.e.



OR

$|x-(-1)| \le 3$

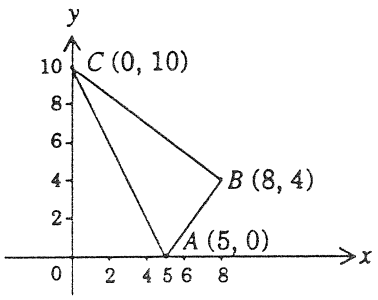
$\therefore \quad -3 \le x + 1 \le 3$

$-4 \le x \le 2$



QUESTION TWO

(a)



(b) Gradient of $AB = \dfrac{4-0}{8-5} = \dfrac{4}{3}$

$y - 0 = \dfrac{4}{3}(x - 5)$

$3y = 4x - 20$

(c) Gradient of $BC = \dfrac{10-4}{0-8} = -\dfrac{3}{4}$

(d) Gradient of $AB \times$ gradient of BC

$= \dfrac{4}{3} \times -\dfrac{3}{4} = -1$

i.e. $m_1 m_2 = -1$

$\therefore$ AB and BC are perpendicular

(e) $AB = \sqrt{(8-5)^2 + (4-0)^2}$ units

$= \sqrt{25}$ units $= 5$ units

(f) In triangles ACO, ACB: AC is common

$\angle COA = 90°$ (coordinate axes meet at angle of 90°)

$\angle ABC = 90°$ (AB and BC are perpendicular)

$A = 5$ units $= AB$

$\therefore \quad \Delta ACO \equiv \Delta ACB$ (RHS)

(g) Area $\Delta OAC = \frac{1}{2} \times 5 \times 10 = 25$ unit^2

$\therefore$ Area quadrilateral $ABCO$

$= 50$ unit^2 (quad. = 2 cgt. Δs)

(h) D (8, 0) and AB: $4x - 3y - 20 = 0$

$$d = \left| \frac{4 \times 8 - 3 \times 0 - 20}{\sqrt{4^2 + (-3)^2}} \right|$$

$$= \frac{12}{5} \text{ units} = 2\frac{2}{5} \text{ units}$$

QUESTION THREE

(a) (i) $\frac{d}{dx}(3x^2 + 5x) = 6x + 5$

(ii) $\frac{d}{dx}(\tan 3x) = \sec^2 3x \times \frac{d}{dx}(3x)$

$= \sec^2 3x \times 3$

$= 3 \sec^2 3x$

(iii) Let $y = \frac{\ln x}{x}$

$u = \ln x \qquad v = x$

$\frac{du}{dx} = \frac{1}{x} \qquad \frac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{x \times \frac{1}{x} - \ln x \times 1}{x^2}$$

$$= \frac{1 - \ln x}{x^2}$$

(b) $\int \frac{dx}{3x+5} = \frac{1}{3}\int \frac{3dx}{3x+5}$

$= \frac{1}{3}\ln(3x+5) + c$

(c) $\int_0^1 e^{4x}\,dx = \frac{1}{4}\left[e^{4x}\right]_0^1$

$= \frac{1}{4}\left(e^4 - e^0\right)^0$

$= \frac{1}{4}\left(e^4 - 1\right)$

(d)

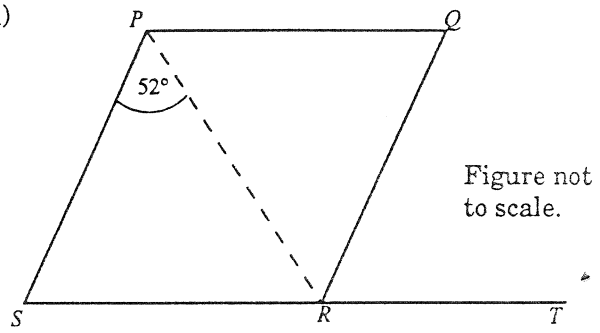


Figure not to scale.

(i) $\angle SPQ = 52° \times 2$ (diagonals of a rhombus bisect the angles through which they pass)

$= 104°$

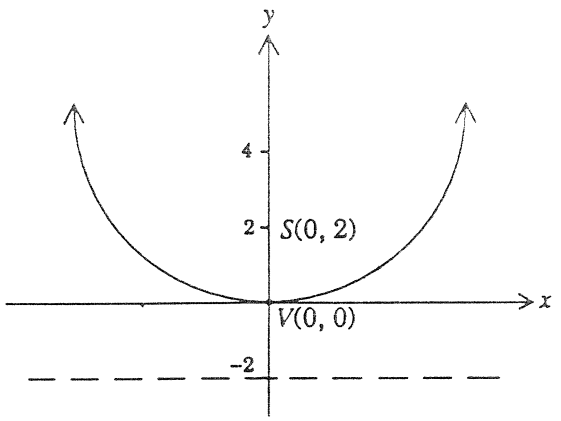
(ii) $\angle QRS = 104°$ (opposite angles of a rhombus equal)

$\angle QRT = 180° - \angle QRS$ ($\angle SRT$ a straight angle)

$= 76°$

QUESTION FOUR

(a) (i) Vertex V is at the origin, i.e. at (0, 0).



(ii) Focal length

= distance from vertex to focus

= distance from vertex to directrix

= 2 units.

(iii) $a = 2$ from (ii)

$\therefore \quad x^2 = 4ay$ becomes $x^2 = 8y$.

(b) (i)

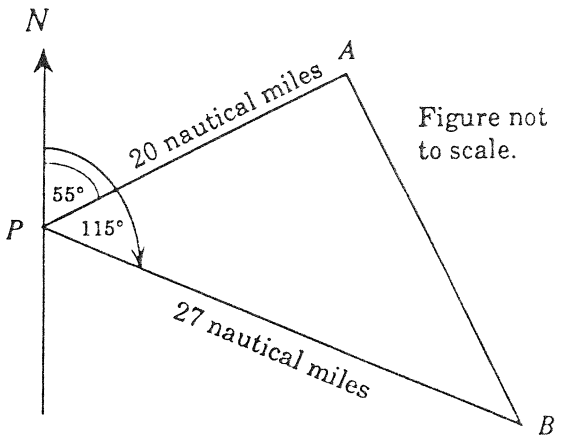


Figure not to scale.

(ii) $\angle APB = \angle NPB - \angle NPA$
$= 115° - 55°$
$= 60°$

(iii) $AB^2 = 20^2 + 27^2 - 2 \times 20 \times 27 \times \cos 60°$
$= 589$
$\therefore AB = \sqrt{589}$
Distance = 24·3 nautical miles (to 1 dec. pl.)

(c) (i) P (both alive) $= 0{\cdot}8 \times 0{\cdot}7 = 0{\cdot}56$

(ii) P (only one alive)
$= P$ (woman alive, husband dead)
$+ P$ (woman dead, husband alive)
$\therefore P$ (only one alive)
$= 0{\cdot}8 \times 0{\cdot}3 + 0{\cdot}2 \times 0{\cdot}7 = 0{\cdot}38$

QUESTION FIVE

(a) $a = 1, \quad b = -5, \quad c = 2.$

(i) $\alpha + \beta = -\frac{b}{a}$
$= \frac{5}{1} = 5$

(ii) $\alpha\beta = \frac{c}{a} = 2$

(iii) $(\alpha + 1)(\beta + 1) = \alpha\beta + \alpha + \beta + 1$
$= 2 + 5 + 1 = 8$

(b) $T_{10} = a + 9d = 29 \quad \ldots(1)$
$T_{15} = a + 14d = 44 \quad \ldots(2)$

(i) (1) − (2): $-5d = -15$
$\therefore d = 3$
Substitute for $d = 3$ in (1):
$a + 27 = 29$
$a = 2$
i.e. common difference is 3 and first term is 2.

(ii) $n = 75, \quad a = 2, \quad d = 3.$
$S_{75} = \frac{75}{2}[2 \times 2 + 74 \times 3]$—using
$S_n = \frac{n}{2}[29 + (n-1)d]$
$= 8475$
i.e. sum of the first 75 terms is 8475.

(c) (i) 45 kilometres in 1 hour
= 45 kilometres in 60 minutes
= 45/60 kilometres in 1 minute
i.e. l km = 3/4 km (or 0·75 km)

(ii) Using $l = r\theta$:
$\frac{3}{4} = 0{\cdot}5 \times \theta$
$\therefore \quad \theta = 1{\cdot}5$ radians
$= 1{\cdot}5 \times \frac{180°}{\pi}$
$\therefore \quad \theta \equiv 86°$ (nearest degree)

QUESTION SIX

(a) $y = 3x^2 - x^3$

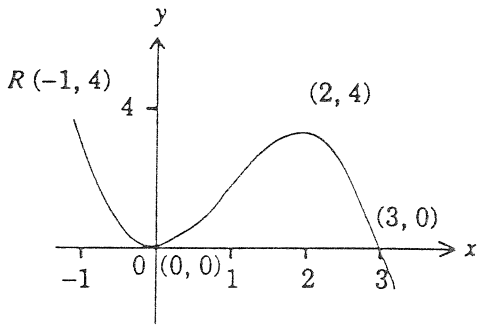
(i) $\frac{dy}{dx} = 6x - 3x^2 = 0$ for stationary points.
$6x - 3x^2 = 0$
$2x - x^2 = 0$
$x(2 - x) = 0$
$\therefore \quad x = 0$ or $2.$

$\frac{d^2y}{dx^2} = 6 - 6x$

$x = 0, \quad \frac{d^2y}{dx^2} > 0.$
$\therefore$ minimum at (0, 0).

$x = 2, \quad \frac{d^2y}{dx^2} < 0.$
$\therefore$ maximum at (2, 4).

(ii) Curve crosses x-axis where $y = 0$.
$\therefore \quad 3x^2 - x^3 = 0$
$x^2(3 - x) = 0$
$x = 0$ or $3.$



(iii) $y = 3x^2 - x^3$
$\frac{dy}{dx} = 6x - 3x^2$
$x = -1, \quad \frac{dy}{dx} = -9$
$y - 4 = -9(x + 1)$
$y - 4 = -9x - 9$
$y = -9x - 5 \quad$ or $\quad 9x + y + 5 = 0.$

(b) (i)

x	0	40	80	120	160
$y = f(x)$	180	120	150	60	120

(ii) $A \cong \frac{80-0}{6}(180 + 150 + 4 \times 120)$

$+ \frac{160-80}{6}(150 + 120 + 4 \times 60)$

$A \cong 1320 \times \frac{80}{6}$

Area $\cong 17\,600\ m^2$

QUESTION SEVEN

(a) (i) $\angle ADE = \angle ABC$ (corresp. angles equal, $DE \parallel BC$)

$\angle AED = \angle ACB$ (corresp. angles equal, $DE \parallel BC$)

$\therefore$ In ΔADE, $\angle ADE = \angle AED$ (ΔABC isosceles so that $\angle ABC = \angle ACB$)

$\therefore$ ΔADE isosceles (2 angles equal)

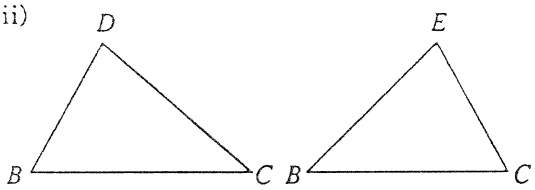
(ii) $AB = AC$ (given)

$AD = AE$ (equal sides oppos. equal angles in isosceles ΔADE)

$\therefore$ $AB - AD = AC - AE$

$\therefore$ $DB = EC$

(iii)



BC is common.

$DB = EC$ [from (ii)]

$\angle DBC = \angle ECB$ (ΔABC isosceles so that angles opposite equal sides are equal)

$\therefore$ $\Delta DBC \equiv \Delta ECB$ (SAS)

(b) (i) $A = A_0 e^{-kt}$ ...(1)

$\frac{dA}{dt} = A_0 \times -ke^{-kt}$

$= -k \times A_0 e^{-kt}$

$\therefore$ $\frac{dA}{dt} = -kA$. from (1)

(ii) $t = 0$, $A = A_0 \times 1$

i.e. $t = 0$, $A = A_0$.

$\therefore$ $t = 5500$, $A = \frac{1}{2}A_0$

$\therefore$ $\frac{1}{2}A_0 = A_0 e^{-5500k}$

$\therefore$ $e^{-5500k} = \frac{1}{2}$

$-5500k = \log_e \frac{1}{2}$

$\therefore$ $k \cong 1{\cdot}26 \times 10^{-4}$ or $0{\cdot}000\,126$

(iii) $A = 0{\cdot}15A_0$, $t = ?$

$0{\cdot}15A_0 = A_0 e^{-1{\cdot}26 \times 10^{-4} \times t}$

$e^{-1{\cdot}26 \times 10^{-4} \times t} = 0{\cdot}15$

$\therefore$ $t = 15\,056{\cdot}5$

i.e. approx. 15 000 years.

QUESTION EIGHT

(a) $V = \pi \int_0^2 y^2\,dx$

where $y^2 = (3x+1)^2$

$= 9x^2 + 6x + 1$

$\therefore$ $V = \pi \int_0^2 (9x^2 + 6x + 1)\,dx$

$= \pi \left[\frac{9}{3}x^3 + \frac{6}{2}x^2 + x\right]_0^2$

$= \pi \left[3x^3 + 3x^2 + x\right]_0^2$

$= \pi(3 \times 2^3 + 3 \times 2^2 + 2) - \pi \times 0$

$\therefore$ Volume $= 38\pi$ units^3

($\cong 119{\cdot}4$ units^3, to 1 dec. pl.)

(b) $x = 1 - 2\sin 2t$

(i) $t = 0$, $x = 1 - 2\sin 0 = 1$

i.e. 1 metre on right-hand side of 0.

(ii) $V = \frac{dx}{dt} = -4\cos 2t$

'Comes to rest' means '$v = 0$'.

$\therefore$ $-4\cos 2t = 0$

$\cos 2t = 0$

$\therefore$ $2t = \frac{\pi}{2}$ (first time)

$t = \frac{\pi}{4}$.

i.e. first comes to rest after $\frac{\pi}{4}$ seconds.

$t = \frac{\pi}{4}$, $x = 1 - 2\sin^2 \frac{\pi}{4}$

$= 1 - 2\sin \frac{\pi}{2} = 1 - 2 = -1$

i.e. first comes to rest 1 metre on the left-hand side of 0.

(iii) $V = 0$

$$\cos 2t = 0$$

$$2t = \frac{\pi}{2}, \quad \frac{3\pi}{2}.$$

$$\therefore \quad t = \frac{\pi}{4}, \quad \frac{3\pi}{4}.$$

$$t = \frac{3\pi}{4},$$

$$x = 1 - 2\sin\left(2 \times \frac{3\pi}{4}\right)$$

$$= 1 - 2\sin\frac{3\pi}{2}$$

$$= 1 - 2 \times -1$$

$$= 1 + 2 = 3$$

i.e. 3 metres on right-hand side of 0.

(iv) $a = \frac{dv}{dt} = \frac{d}{dt}(-4\cos 2t)$

$$= 8\sin 2t$$

$$t = \frac{\pi}{12}, \quad a = 8\sin\left(2\frac{\pi}{12}\right)$$

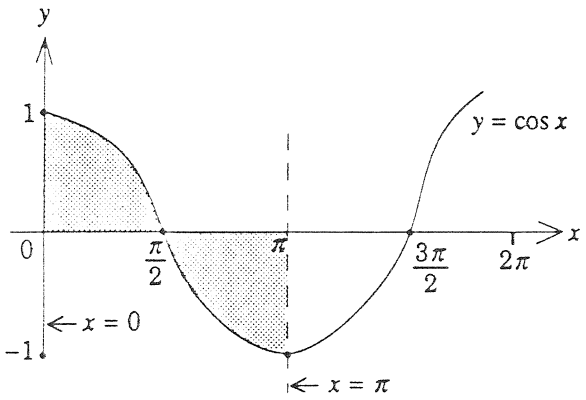
$$= 8\sin\frac{\pi}{6}$$

$$= 8 \times \frac{1}{2} = 4$$

i.e. acceleration is 4 ms^{-2}.

QUESTION NINE

(a) (i)



(ii) See above, for shaded regions.
Either:

$$A = \int_0^{\frac{\pi}{2}} \cos x\,dx + \left| \int_{\frac{\pi}{2}}^{\pi} \cos x\,dx \right|$$

$$= [\sin x]_0^{\frac{\pi}{2}} + \left| [\sin x]_{\frac{\pi}{2}}^{\pi} \right|$$

$$= \left(\sin\frac{\pi}{2} - \sin 0\right) + \left|\left(\sin\pi - \sin\frac{\pi}{2}\right)\right|$$

$$= 1 - 0 + |0 - 1|$$

$$= 1 + 1$$

$\therefore$ Area 2 unit^2.

Or:

$$A = 2\int_0^{\frac{\pi}{2}} \cos x\,dx$$

$$= 2[\sin x]_0^{\frac{\pi}{2}}$$

$$= 2\left(\sin\frac{\pi}{2} - \sin 0\right)$$

$$= 2(1 - 0)$$

$\therefore$ Area 2 unit^2.

(iii) $\cos x = \frac{\sqrt{3}}{2}, \qquad 0 \le x \le 2\pi$

$$\therefore \quad x = \frac{\pi}{6}, \quad 2\pi - \frac{\pi}{6}$$

$$\therefore \quad x = \frac{\pi}{6}, \quad \frac{11\pi}{6}$$

(b) (i) $r = 5$ and $n = 100$

$$\therefore \quad R = 1 + \frac{r}{100}$$

$$= 1 + \frac{5}{100}$$

$$= 1{\cdot}05$$

Using $A = PR^n$ where $P = 1000$;

$$A = 1000 \times (1{\cdot}05)^{100}$$

$$= 131\,501{\cdot}26$$

i.e. \$131 501·26

(ii) After 200 years, the initial \$1000 investment will become

$$\$1000 \times (1{\cdot}05)^{200}.$$

At the beginning of the second year, another \$1000 is invested for 199 years. This amount will become

$$\$1000 \times (1{\cdot}05)^{199}.$$

At the beginning of the 200th year, the last \$1000 will be invested for one year. This amount will become

$$\$1000 \times 1{\cdot}05.$$

$\therefore$ Total amount in fund after 200 years

$$= \$(1000 \times 1{\cdot}05^1 + 1000 \times 1{\cdot}05^2 + \ldots$$
$$+ 1000 \times 1{\cdot}05^{199} + 100 \times 1{\cdot}05^{200})$$

$$= \$1000\,(1{\cdot}05^1 + 1{\cdot}05^2 + \ldots$$
$$+ 1{\cdot}05^{99} + 1{\cdot}05^{200})$$

$$= \$1000 \times \frac{a(r^n - 1)}{r - 1} \quad \text{where} \quad \begin{matrix} a = 1{\cdot}05 \\ n = 200 \\ r = 1{\cdot}05 \end{matrix}$$

$$= \$1000 \times \frac{1{\cdot}05(1{\cdot}05^{200} - 1)}{1{\cdot}05 - 1}$$

$$= \$363\,123\,196{\cdot}90.$$

QUESTION TEN

(a) $x^2 + (k+2)x + 4 = 0$

$a = 1, \quad b = k+2, \quad c = 4.$

$$\Delta = b^2 - 4ac$$
$$= (k+2)^2 - 4 \times 1 \times 4$$
$$= k^2 + 4k - 12$$

(i) For equal roots, $\Delta = 0$.

$$k^2 + 4k - 12 = 0$$
$$(k+6)(k-2) = 0$$
$$k = -6 \quad \text{or} \quad 2.$$

(ii) For distinct real roots, $\Delta = 0$.

$$(k+6)(k-2) > 0$$
$$k < -6 \quad \text{or} \quad k > 2.$$

(b) (i) $OP^2 = x^2 + 1$

$OP = \sqrt{x^2+1}$ km

$\therefore$ Time to travel distance OP

$$= \frac{\text{distance travelled}}{\text{speed}}$$
$$= \frac{\sqrt{x^2+1}}{6} h.$$

$PB = (2 - x)$ km.

$\therefore$ Time to travel distance PB

$$= \frac{2-x}{10} h.$$

$$\therefore \quad T = \frac{\sqrt{x^2+1}}{6} + \frac{2-x}{10}$$

(ii) $$T = \frac{1}{6}(x^2+1)^{\frac{1}{2}} + \frac{1}{5} - \frac{1}{10}x$$

$$\frac{dT}{dx} = \frac{1}{12} \times 2x(x^2+1)^{-\frac{1}{2}} - \frac{1}{10}$$
$$= \frac{x}{6\sqrt{x^2+1}} - \frac{1}{10}$$

$\frac{dT}{dx} = 0$ for max. or min.

$$\therefore \quad \frac{x}{6\sqrt{x^2+1}} - \frac{1}{10} = 0$$

$$x = \frac{6\sqrt{x^2+1}}{10}$$

Square both sides

$$x^2 = \frac{36(x^2+1)}{100}$$
$$100x^2 = 36x^2 + 36$$
$$64x^2 = 36$$
$$x = \pm\frac{3}{4}$$

Either:

x	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}+$
$\frac{dT}{dx}$	$-$	0	$+$

$\therefore$ Min. when $x = \frac{3}{4}$

i.e. to minimize time, should row to point P, $\frac{3}{4}$ km from A.

Or:

$$\frac{d^2T}{dx^2} = \frac{6\sqrt{x^2+1} - x \times 6 \times \frac{1}{2} \times \frac{2x}{\sqrt{x^2+1}}}{36(x^2+1)}$$
$$= \frac{6\sqrt{x^2+1} - \frac{6x^2}{\sqrt{x^2+1}}}{36(x^2+1)}$$
$$= \frac{6(x^2+1) - 6x^2}{36(x^2+1)\sqrt{x^2+1}}$$
$$= \frac{6}{36(x^2+1)^{\frac{3}{2}}}$$
$$= \frac{1}{6(x^2+1)^{\frac{3}{2}}}$$

> 0 for all real x.

i.e. minimum when $x = \frac{3}{4}$

$\left(\text{since } \frac{d^2T}{dx^2} > 0\right)$.