HIGHER SCHOOL CERTIFICATE MATHEMATICS
1990
2/3 UNIT SOLUTIONS

QUESTION ONE

(a) $\dfrac{171\cdot8}{14\cdot3\times5\cdot7}$

$=2\cdot1077$

$=2.11$ correct to two decimal places.

(b) $299\ 725\text{ km/s} = 2\cdot997\times10^5\text{ km/s}$

correct to four significant figures.

(c) $2x^2-7x-15=(2x+3)(x-5)$

(d) $V=\frac{4}{3}\pi r^3$

$r=\sqrt[3]{\frac{3V}{4\pi}}$

$=1\cdot0607\cdots$

$=1\cdot06$ cm correct to two decimal places.

(e) $3-(4-x)=5x$

$3-4+x=5x$

$-1=4x$

$-\frac{1}{4}=x$

$x=-\frac{1}{4}$

(f) $|x-2|<3$

$x-2<3$ or $-(x-2)<3$

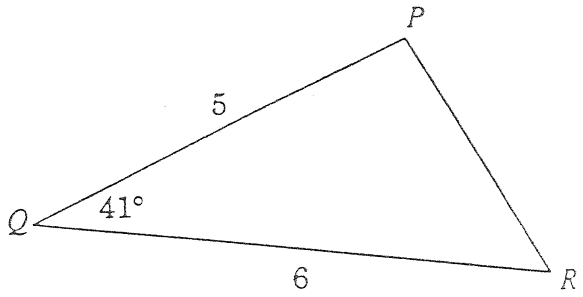
$x<5$ or $x-2>-3$

$x>-1$

$\therefore\ -1<x<5.$



QUESTION TWO



(a) $PR^2=QP^2+QR^2-2QP.QR.\cos41°$

$=5^2+6^2-2.5.6.\cos41°$

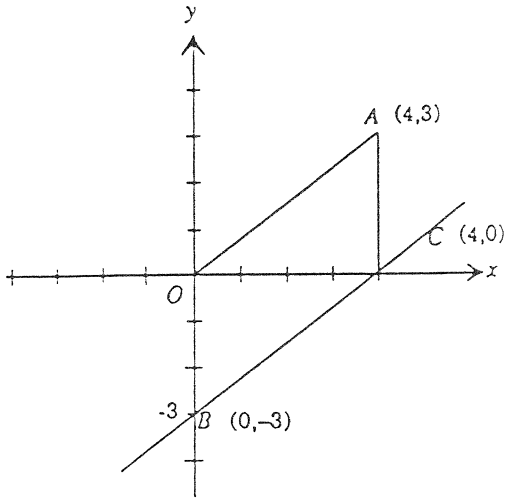
$=15\cdot7174\cdots$

$PR=3\cdot9645\cdots$

$=3\cdot96$ correct to two decimal places.

(b) (i)

(ii)



(iii) Equation of BC

$\dfrac{y-(-3)}{x-0}=\dfrac{0-(-3)}{4-0}$

$\dfrac{y+3}{x}=\dfrac{3}{4}$

$4y+12=3x$

$4y=3x-12$

(iv) Gradient of $OA = \frac{3-0}{4-0}$

$= \frac{3}{4}$

BC has gradient $\frac{3}{4}$ [from part (iii)]

So $OA \| BC$

OB lies on the y axis (i.e. the line $x = 0$)

AC lies on the line $x = 4$ and so is parallel to the y axis

So $OB \| AC$.

Since the opposite sides of $OACB$ are parallel, $OACB$ is a parallelogram.

(v) Area of $OACB$ = Base $AC \times$ perp height CO

$= 3 \times 4$

$= 12$ units^2

(vi) $AB^2 = (4-0)^2 + (3-(-3))^2$

$= 16 + 36$

$= 52$

$AB = 2\sqrt{13}$ units

QUESTION THREE

(a) (i) $\frac{2}{2-\sqrt{3}} + \frac{2+\sqrt{3}}{2+\sqrt{3}} = \frac{4+2\sqrt{3}}{4-3}$

$= 4 + 2\sqrt{3}$

(ii) $a + \sqrt{b} = 4 + 2\sqrt{3}$

$a = 4, \sqrt{b} = 2\sqrt{3} = \sqrt{12}$

$b = 12$

(b) (i) $\frac{d}{dx}(7x^5 + 3) = 35x^4$

(ii) $\frac{d}{dx}(2\sin x + \sqrt{x}) = 2\cos x + \frac{1}{2\sqrt{x}}$

(iii) $\frac{d}{dx}(x + \ln x) = \ln x . \frac{d}{dx}(x) + x \frac{d}{dx}(\ln x)$

$= \ln x . 1 + x . \frac{1}{x}$

$= \ln x + 1$

(c) (i) $\int (\cos x + 5x^2)\,dx = \sin x + \frac{5x^3}{3} + C$

(ii) $\int_0^1 (1 + e^{-x})\,dx = \left[x - e^{-x}\right]_0^1$

$= (1 - e^{-1}) - (o - e^{o})$

$= 1 - \frac{1}{e} + 1$

$= 2 - \frac{1}{e}$

QUESTION FOUR

(a) $S_n = \frac{n(3n+1)}{2}$

(i) $S_1 = \frac{1(3+1)}{2}$

$= 2$

$S_2 = \frac{2(6+1)}{2}$

$= 7$

(ii) $T_1 = S_1 = 2 = a$

$T_2 = S_2 - S_1 = 5 = a + d$

so $d = 3$

$T_3 = a + 2d = 8$

$T_1 = 2,\ T_2 = 5,\ T_3 = 8$

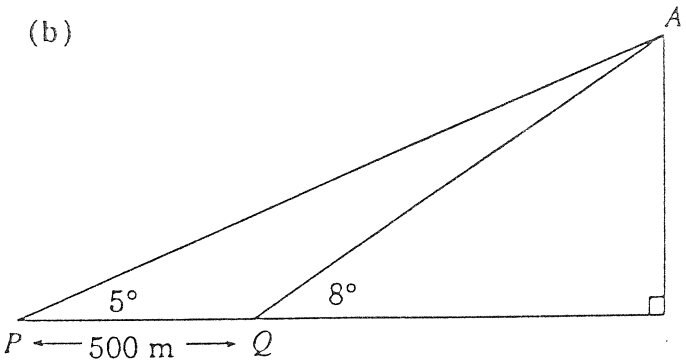
(iii) $T_n = a + (n-1)d$

$= 2 + (n-1).3$

$= 2 + 3n - 3$

$= 3n - 1$

(b)



(i) $\angle PAQ = 3°$

(ii) $\dfrac{AQ}{\sin\angle P} = \dfrac{PQ}{\sin\angle PAQ}$

$\dfrac{AQ}{\sin 5^\circ} = \dfrac{500}{\sin 3^\circ}$

$AQ = \dfrac{500\sin 5^\circ}{\sin 3^\circ}$

$= 833$ m to the nearest metre.

(iii) $\dfrac{QB}{AQ} = \cos\angle AQB$

$QB = AQ.\cos\angle AQB$

$= 833.\cos 8^\circ$

$= 820$ m to nearest 10 m.

QUESTION FIVE

(a) $y = 1+3x-x^3$, for $-2 \le x \le 3$

(i) $\dfrac{dy}{dx} = 3-3x^2$

Stationary points occur when $\dfrac{dy}{dx} = 0$

$3-3x^2 = 0$

$x = 1$ or -1

$y = 3$ or -1

$\dfrac{d^2y}{dx^2} = -6x$

At $x = 1$, $\dfrac{d^2y}{dx^2} = -6$

so there is a maximum at (1,3)

At $x = -1$, $\dfrac{d^2y}{dx^2} = -6$

so there is a minimum at $(-1,-1)$.

(ii) Point of inflexion occurs when $\dfrac{d^2y}{dx^2} = 0$

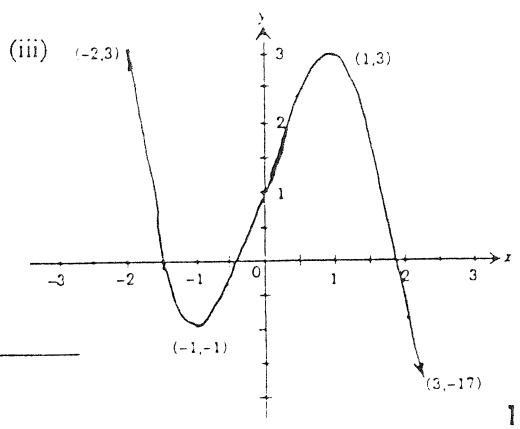
$-6x = 0$

$x = 0$

$y = 1$

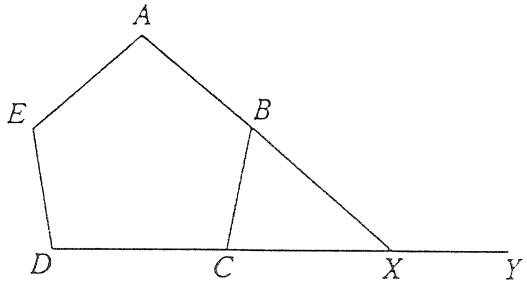
Point of inflexion is (0, 1).

(iii)



(iv) Minimum value of $1+3x-x^3$ for $-2 \le x \le 3$ occurs when $x = 3$ and that value is -17.

(b)



(i) $\angle ABC = \dfrac{3\times 180}{5}$

$= 108^\circ$

(ii) $\angle ABC = \angle BCD = 108^\circ$ (internal angles of regular pentagon)

$\angle XBC = \angle XCB = 72^\circ$ (supplementary angles)

$\angle BXY = \angle XBC + \angle XCB$ (exterior angle of triangle is equal to the sum of the interior opposite angles)

$= 72^\circ+72^\circ$

$= 144^\circ$.

QUESTION SIX

(a) $\tan x = \dfrac{1}{\sqrt{3}}$

$x = 30^\circ$ and 210°

(b) (i) $v = \dfrac{1}{t+1}$

$a = \dfrac{dv}{dt} = -\dfrac{1}{(t+1)^2}$.

When $t = 1$,

$a = -\dfrac{1}{(1+1)^2}$

$= -\dfrac{1}{4}$

(ii) $x = \int_0^1 \left(\frac{1}{t+1}\right) dt$

$= [\ln(t+1)]_0^1$

$= \ln 2 - \ln 1$

$= \ln 2$

(c) TABLE OF MAXIMUM FACE VALUES

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	2	3	4	5	6
3	3	3	3	4	5	6
4	4	4	4	4	5	6
5	5	5	5	5	5	6
6	6	6	6	6	6	6

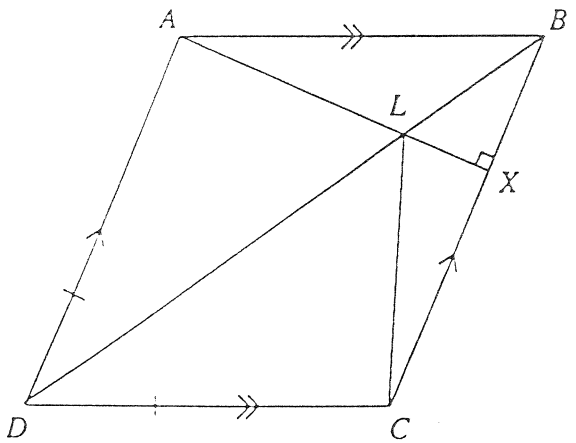
(i) $P(1) = \frac{1}{36}$

(ii) $P(1,1,1) = \frac{1}{36} \times \frac{1}{36} \times \frac{1}{36} = \frac{1}{46656}$

(iii) $P(6) = \frac{11}{36}$

QUESTION SEVEN

(a)



(i) $\angle ADB = \angle CDB$ (since the diagonals of the rhombus bisect the opposite vertices)

(ii) In Δs ALD, CLD

$AD = CD$ (adjacent sides of rhombus equal)

$\angle ADL = \angle CDL$ [Part (i)]

DL is common

Hence Δs ALD, CLD are congruent (S.A.S.).

(iii) $\angle BXA = \angle XAD$ are alternate angles

Since $AD \| BC$ (opposite sides of rhombus are parallel)

and $\angle BXA$ is a right angle (given)

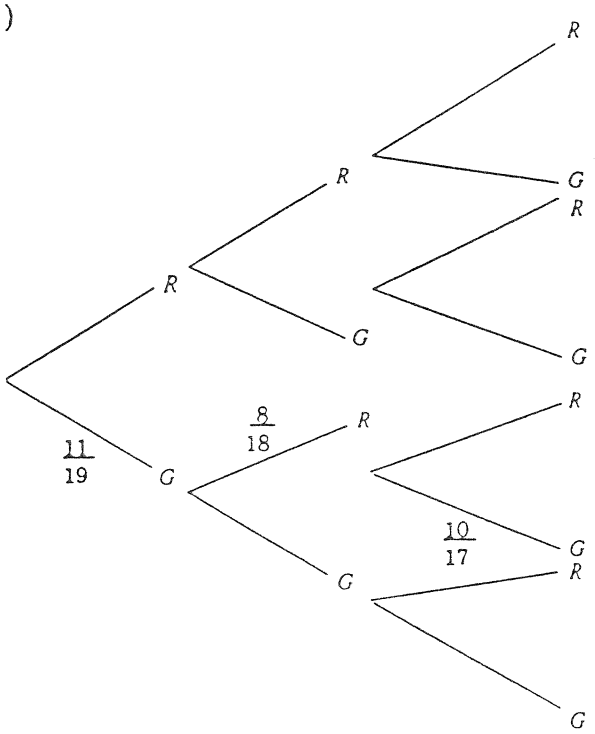
then $\angle XAD = \angle BXA = 90°$.

(iv) Since Δs ALD, CLD are congruent [part (ii)]

$\angle DCL = \angle DAL = 90°$

$\therefore$ $\angle LCD$ is a right angle.

(b)



$P(G,R,G) = \frac{11}{19} \times \frac{8}{18} \times \frac{10}{17}$

$= \frac{440}{2907}$

(c) $Q = Q_0 e^{-kt}$

(i) $\frac{1}{2}Q_0 = Q_0 e^{-k.1690}$

$\frac{1}{2} = e^{-1690k.}$

$\ln\frac{1}{2} = -1690k.$

$\frac{\ln\frac{1}{2}}{-1690} = k$

$4\cdot1014\times10^{-4} = k$

$k = 4\cdot1014\times10^{-4}$

(ii) When $Q = 20\%$ of Q_0

$\frac{1}{5}Q_0 = Q_0 e^{-kt}$

$\ln\frac{1}{5} = -kt$

$\frac{\ln\frac{1}{5}}{-k} = t$

$3924\cdot05\cdots = t$

After 3924 years 20% of the initial amount of radium remains.

QUESTION EIGHT

(a) $y = x^2 + x$

$\frac{dy}{dx} = 2x + 1$

At $x = 1$, gradient of tangent is 3

and gradient of normal is $-\frac{1}{3}$.

Also at $x = 1$, $y = 2$.

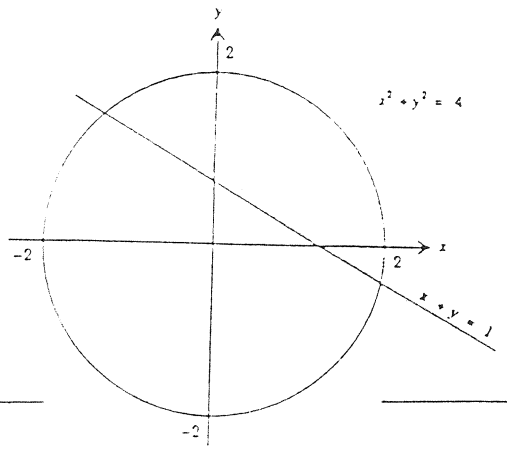
Equation of normal

$\frac{y-2}{x-1} = -\frac{1}{3}$

$3y - 6 = -x + 1$

$x + 3y - 7 = 0$ is required equation.

(b)



(c) $N(t) = \frac{A}{2+e^{-t}}$

(i) $N(0) = 3\times10^5$

$3\times10^5 = \frac{A}{2+e^0}$

$3\times10^5 = \frac{A}{2+1}$

$9\times10^5 = A.$

The value of A is 9×10^5.

(ii) $N(1) = \frac{A}{2+e^1}$

$= \frac{9\times10^5}{2+\frac{1}{e}}$

$= 380087$ termites.

(iii) When t is very large e^{-t} is nearly zero. The number of termites would be close to

$\frac{9\times10^5}{2+0}$

i.e. close to 380087 termites.

(iv) $\frac{dN}{dt} = A.-1(2+e^{-t})^{-2}.-e^{-t}$

$= \frac{Ae^{-t}}{\left(2+e^{-t}\right)^2}$

QUESTION NINE

(a) $y = \sin^2 x$

(i)

x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
y	0	$1/2$	1	$1/2$	0

(ii) $\int_0^{\pi} \sin^2 x\,dx$

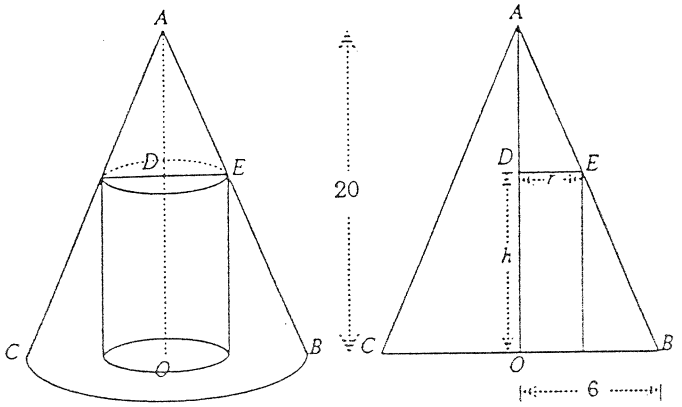
$= \frac{h}{3}\left[f(0) + 4f(\frac{\pi}{4}) + 2f(\frac{\pi}{2}) + 4f(\frac{3\pi}{4}) + f(\pi)\right]$

$= \frac{\pi}{4.3}\left[0 + 4.\frac{1}{2} + 2.1 + 4.\frac{1}{2} + 0\right]$

$= \frac{\pi}{12}\times6$

$= \frac{\pi}{2}$

(b)



Take a cross section of the cone from the vertex A to the diameter CB at the base of the cone.

Since DE is parallel to OB, triangles ADE and AOB are similar.

So $\dfrac{AD}{AO} = \dfrac{DE}{OB}$

$$\frac{20-h}{20} = \frac{r}{6}$$

$$6.20 - 6h = 20r$$

$$6.20 - 20r = 6h$$

$$\frac{10(6-r)}{3} = h.$$

(i) $V = \pi r^2 h$

$$= \pi r^2 . \frac{10(6-r)}{3}$$

$$= \frac{10\pi r^2 (6-r)}{3}$$

(ii) $$\frac{dV}{dr} = \frac{10\pi}{3}\left(12r - 3r^2\right)$$

$$= 10\pi\left(4r - r^2\right)$$

There is a maximum volume when

$$\frac{dV}{dr} = 0 \text{ and } \frac{d^2V}{dr^2} < 0$$

$$10\pi\left(4r - r^2\right) = 0$$

$$r = 0 \text{ or } r = 4$$

$$\frac{d^2V}{dr^2} = 10\pi(4 - 2r)$$

$$= 10\pi(4-8)$$

$$= -40\pi \text{ when } r = 4.$$

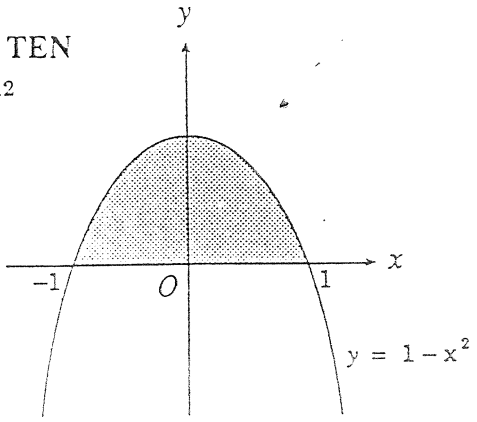
So there is a maximum volume when $r = 4$ and $h = \frac{20}{3}$.

(iii) Max. volume $= \pi . 4^2 . \frac{20}{3}$

$$= \frac{320\pi}{3} \text{ units}^3.$$

QUESTION TEN

(a) $y = 1 - x^2$



When $y = 0$, $1 - x^2 = 0$ and $x = 1$ or -1

$$V = \int_{-1}^{1} \pi y^2 dx$$

$$= \int_{-1}^{1} \pi\left(1 - x^2\right)^2 dx$$

$$= \pi \int_{-1}^{1} \left(1 - 2x^2 + x^4\right) dx$$

$$= \pi\left[x - \frac{2}{3}x^3 + \frac{x^5}{5}\right]_{-1}^{1}$$

$$= \pi\left[\frac{8}{15} - \left(\frac{-8}{15}\right)\right]$$

$$= \frac{16\pi}{15} \text{ units}^3.$$

(b) A_n is the amount owed at the end of n months and P is the amount borrowed.

(i) $A_1 = P + \frac{2}{100}P$

$= 1{\cdot}02P \qquad P = 80\,000$

$= 1{\cdot}02 \times 80\,000$

$= \$81\,600.$

(ii) $A_2 = A_1 + \frac{2}{100}A_1$

$= 1{\cdot}02A_1$

$= (1{\cdot}02)^2 P$

$A_3 = (1{\cdot}02)A_2$

$= (1{\cdot}02)^3 P$

So $A_{12} = (1{\cdot}02)^{12} P$

$= \$80\,000 \times (1{\cdot}02)^{12}.$

When the first instalment of SM is paid the total amount owed is $80\ 000\times(1\cdot02)^{12}-M$.

(iii)

$$A_{13}=\left[(1\cdot02)^{12}\times P-M\right]+\frac{2}{100}\left[(1\cdot02)^{12}\times P-M\right]$$

$$=1\cdot02\left[(1\cdot02)^{12}\times P-M\right]$$

$$A_{24}=1\cdot02^{12}\left[(1.02)^{12}\times P-M\right]$$

When the final instalment of SM is paid the amount owed is

$$(1\cdot02)^{12}\left[80\ 000(1\cdot02)^{12}-M\right]-M.$$

But the final amount owed is zero.

So $0=(1\cdot02)^{12}\left[80000(1.02)^{12}-M\right]-M$

$$=80000(1\cdot02)^{24}-(1.02)^{12}M-M$$

$$=80000(1\cdot02)^{24}-M\left[(1.02)^{12}+1\right].$$

Hence $M\left[(1.02)^{12}+1\right]$

$$=80000(1\cdot02)^{24}$$

$$M=\frac{80000(1\cdot02)^{24}}{(1.02)^{12}+1}.$$

(iv) Each instalment is

M = S56728.95

Total repaid is

S113459.90

Interest = Total repayment less loan

= S33457.90