HIGHER SCHOOL CERTIFICATE EXAMINATION 1989
MATHEMATICS - 2/3 UNIT COMMON PAPER

QUESTION ONE

(a) (i) $(2.7)^5 \approx 143.49$

(ii) $\frac{-1.38}{10.58} \approx -0.12$

(b) $S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{103} = \frac{103}{2}[2 \times 5 + (103 - 1) \times 1.2]$

$= 6818.6$

(c) Midpoint $(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$= \left(\frac{2 + -4}{2}, \frac{5 + 4}{2}\right)$

$= (-1, 4\frac{1}{2})$

(d) $4(x - 5) = 3 - 2(x - 1)$

$4x - 20 = 3 - 2x + 2$

$6x - 20 = 5$

$6x = 25$

$x = 4\frac{1}{6}$

(e) Sale price = 85% of marked price

$= 0.85 \times \$79.95$

$= \$67.96$

Alternatively

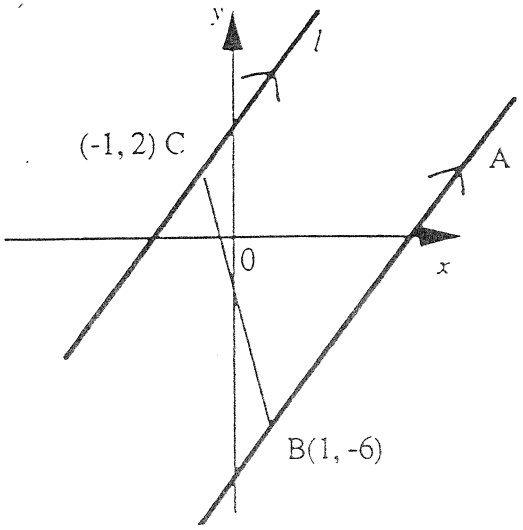
Discount $= 0.15 \times \$79.95$

$= \$11.99$

$\therefore$ sale price $= \$79.95 - \$11.99$

$= \$67.96$

QUESTION TWO



(ii) $d = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$

$BC = \sqrt{\left(1 - (-1)\right)^2 + \left(-6 - 2\right)^2}$

$= \sqrt{4 + 64}$

$= \sqrt{68}$ or $2\sqrt{17}$ units

(iii) Equation of line l is $y = 2x + 4$

Thus slope of l is 2

Let θ = angle of inclination of l to the positive direction of the x-axis

$\tan\theta = 2$

$\theta = 63^\circ$ (to the nearest degree)

(iv) AB is parallel to l

$\therefore$ the slope of AB is 2.

Also AB passes through B(1,-6)

Using $y - y_1 = m(x - x_1)$

the equation of AB is

$y - (-6) = 2(x - 1)$

$y + 6 = 2x - 2$

$y = 2x - 8$ as required

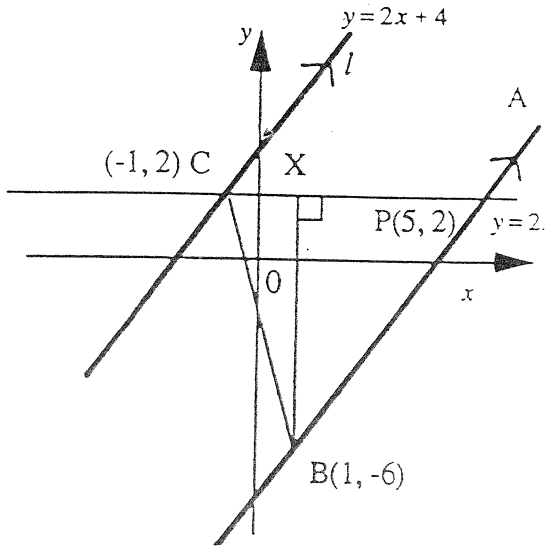
(v) Since P lies on y = 2 and on AB, then at P, y = 2. Substitute y = 2 into the equation of AB to find the x co-ordinate at P

$2 = 2x - 8$

$x = 5$

The co-ordinates of P are (5,2)

(vi) The length of PC = 5 – (-1) = 6 units



(viii) Let X be the foot of the perpendicular from B to the line PC. The perpendicular height of ΔPBC = BX

= 2 – (-6)

= 8 units.

$\therefore$ Area of $\Delta PBC = \frac{1}{2} \times 6 \times 8 = 24$ sq.units

QUESTION THREE

(a)

(i) $\frac{d}{dx}\left(4x^3 - 7 + \frac{5}{x}\right) = \frac{d}{dx}\left(4x^3 - 7 + 5x^{-1}\right)$

$= 12x^2 - 5x^{-2}$

or $12x^2 - \frac{5}{x^2}$

(ii) $\frac{d}{dx}(x \cos x) = x.-\sin x + \cos x.1$

(by the product rule)

$= -x\sin x + \cos x$

(b) $\int \left(x^2 + \frac{2}{x}\right) dx = \frac{x^3}{3} + 2\log_e x + C$

(c) $\int_0^1 \left(e^{3x} + 1\right) dx = \left[\frac{e^{3x}}{3} + x\right]_0^1$

$= \frac{e^3}{3} + 1 - \frac{e^0}{3}$

$= \frac{1}{3}(e^3 + 2)$

(d) a = 1200 d = 150

(i) Assuming calendar months, n = 24

$T_n = a + (n - 1)d$

$T_{24} = 1200 + (24 - 1) \times 150$

$= 4650$

i.e. sales team plans to sell 4650 calculators that month

(ii) $S_n = \frac{n}{2}\left[a + l\right]$ Here $l = T_{24}$

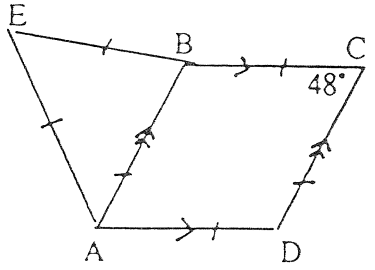
$S_{24} = \frac{24}{2}\left[1200 + 4650\right]$

$= 70200$

i.e. sales team plans to sell 70200 calculators over the entire two year period.

QUESTION FOUR

(i)



(ii) $\angle BAD = 48^\circ$ (opp. angles of rhombus)

$\angle EAB = 60^\circ$ (equilateral Δ)

$\therefore \angle EAD = 48^\circ + 60^\circ = 108^\circ$

(iii) EA = BA (equilateral Δ)

BA = AD (sides of rhombus)

$\therefore$ EA = AD

Thus $\angle DEA = \angle EDA$

(angles opposite equal sides)

$108^\circ + 2\angle EDA = 180^\circ$ (angle sum of Δ)

$\therefore \angle EDA = 36^\circ$

Alternatively

EA = BA (equilateral Δ)

BA = AD (sides of rhombus)

∴ EA = AD

Hence ΔDEA is isosceles

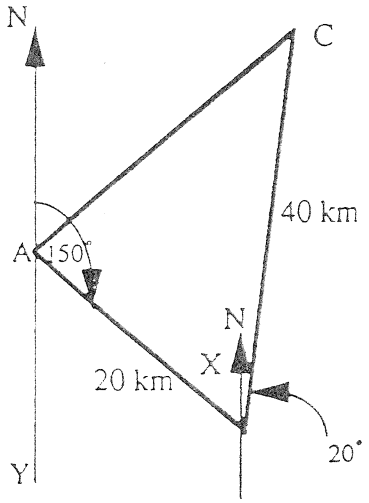
∴ ∠DEA = ∠EDA

(angles opposite equal sides)

$108^\circ + 2\angle EDA = 180^\circ$ (angle sum of Δ)

$\therefore \angle EDA = 36^\circ$

(b)



$\angle ABX + 150^\circ = 180^\circ$

(cointerior angles, parallel lines)

$\therefore \angle ABX = 30^\circ$

Thus $\angle ABC = 30^\circ + 20^\circ = 50^\circ$

Alternatively

$\angle YAB = 30^\circ$ (supplementary angles)

$\therefore \angle ABX = 30^\circ$ (alternate angles, AY//BX)

$\therefore \angle ABC = 30^\circ + 20^\circ = 50^\circ$

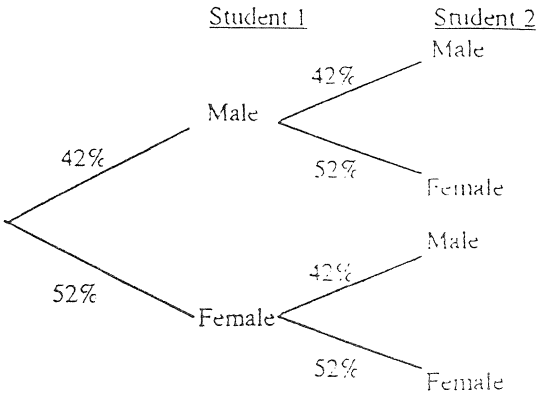
(ii) $AC^2 = AB^2 + BC^2 - 2.AB.BC.\cos B$

$= 20^2 + 40^2 - 2.20.40.\cos 50^\circ$

AC = 31.1695...

AC = 31km (to the nearest km)

(c)



(i) P(female, female) = 0.58 × 0.58

= 0.3364 ≈ 0.34

(ii) P(both of same sex)

= P(female, female) + P(male, male)

= 0.3364 + 0.42 × 0.42

= 0.5128 ≈ 0.51

(iii) P(both of different sexes)

= 1 − P(same sex)

= 1 − 0.5128

= 0.4872 ≈ 0.49

Alternatively

P(both of different sexes)

=P(male, female) +P(female, male)

= 0.42 × 0.58 + 0.58 × 0.42

= 0.4872 ≈ 0.49

QUESTION FIVE

(a) $T_2 = ar = 6$

(i) $T_7 : T_6 = \dfrac{ar^6}{ar^5}$

$= r$

$= 3$

(ii) $T_2 = 6$

$\therefore 3a = 6$

$a = 2$

(iii) $S_n = \dfrac{a\left(r^n - 1\right)}{r-1}$

$$S_{12} = \frac{2\left(3^{12} - 1\right)}{3-1}$$

$$= 531440$$

(b) Volume about x-axis $= \pi \int_a^b y^2 dx$

$$= \pi \int_0^{\frac{\pi}{3}} sec^2 x.dx$$

$$= \pi \left[\tan x\right]_0^{\pi/3}$$

$$= \pi \left[\tan \frac{\pi}{3} - \tan 0\right]$$

$$= \sqrt{3}\,\pi \text{ cubic units}$$

(c) $v = 4 - 2t = \dfrac{dx}{dt}$

$$x = \int \left(4 - 2t\right).\, dt$$

$$= 4t - t^2 + C$$

At $t = 0, x = 1, \therefore C = 1$

so displacement equation is

$x = 4t - t^2 + 1$

(ii) At rest $v = 0$

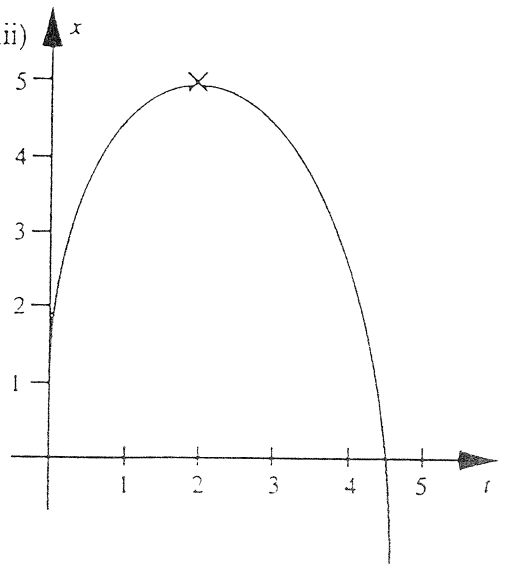
$4 - 2t = 0$

$t = 2$

acceleration $= \ddot{x} = \dfrac{dv}{dt} = -2$

Thus the particle is at rest after 2 secs when acceleration is at -2m s^{-2}

(iii)



(iii) When $t = 2$, $x = 8 - 4 + 1 = 5$. So, in the first two seconds, the particle has travelled $(5 - 1) = 4$ metres. From the graph after 2 seconds the particle changes direction. When $t = 4$, $x = 16 - 16 + 1 = 1$. So, in the next two seconds the particle has travelled a further $|1 - 5| = 4$ metres. Thus the total distance travelled in first 4 seconds is $4 + 4 = 8$m.

QUESTION SIX

(a)

$$4x - y = 3 \quad (1)$$

$$10x + 3y = 2 \quad (2)$$

(1) × 3 $\quad 12x - 3y = 9 \quad (3)$

(2) + (3) $\quad 22x = 11$

$$x = \frac{1}{2}$$

Substitute $x = \frac{1}{2}$ into (1)

$$2 - y = 3$$

$$y = -1$$

Thus $x = \frac{1}{2}$ and $y = -1$

Alternatively

$$4x - y = 3 \quad (1)$$
$$10x + 3y = 2 \quad (2)$$

Take (1) and make y the subject

$$y = 4x - 3$$

Substitute $y = 4x - 3$ into (2)

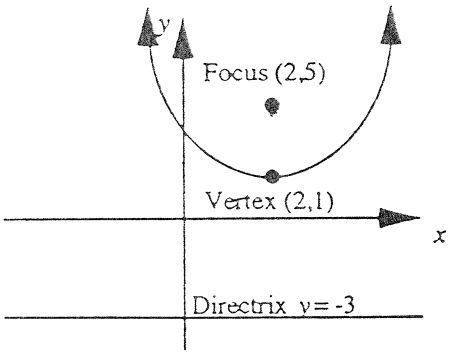
$$10x + 3(4x - 3) = 2$$
$$10x + 12x - 9 = 2$$
$$22x = 11$$
$$x = \frac{1}{2}$$

Substitute $x = \frac{1}{2}$ into (1)

$$2 - y = 3$$
$$y = -1$$

Thus $x = \frac{1}{2}$ and $y = -1$

(b)



Note: The vertex is on the line $x = 2$ (the equation of the axis of symmetry) and half-way between the focus and the directrix.

(ii) Using the general equation of a parabola

$(x-h)^2 = 4A(y-k)$ with

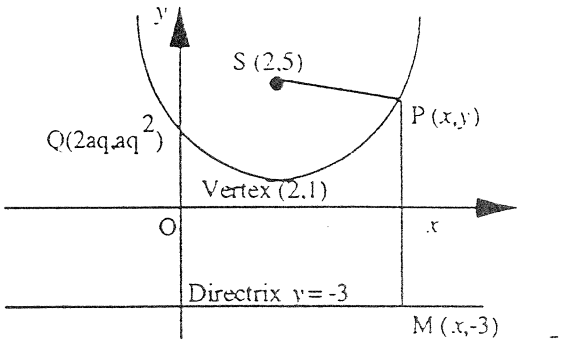
vertex (h,k) = (2, 1) and focal length, A = 4 units,

Equation of parabola is

$$(x-2)^2 = 4.4(y-1)$$

i.e. $(x-2)^2 = 16(y-1)$

Alternatively



Let S be the focus (2,5), P(x,y) a point on the parabola and M$(x, -3)$ a point on the directrix vertically below P. By definition,

$$SP = PM$$
$$SP^2 = PM^2$$

$$(x-2)^2 + (y-5)^2 = (y+3)^2$$
$$x^2 - 4x + 4 + y^2 - 10y + 25 = y^2 + 6y + 9$$
$$x^2 - 4x - 16y + 20 = 0 \text{ is req'd equation}$$

(c) (i) A is point of intersection of $y = 3x^2$ and $y = 4 - x$

Test (1,3) in $y = 3x^2$

$$3 = 3.1^2$$
$$= 3$$

(1,3) satisfies $y = 3x^2$

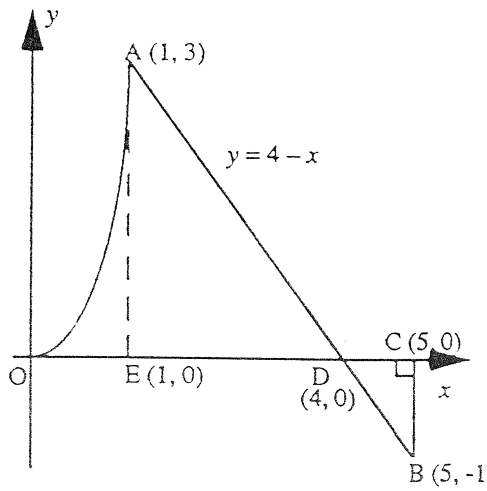
Test (1,3) in $y = 4 - x$

$$3 = 4 - 1$$
$$= 3$$

(1,3) also satisfies $y = 4 - x$

$\therefore$ A has co-ordinates (1,3)

(ii)



Let E have co-ordinates (1,0) . At D the line $y = 4 - x$ cuts the x-axis at i.e. $y = 0$.
Hence $x = 4$ Co-ordinates of D are (4,0)
At B, $x = 5$
Substitute $x = 5$ into $y = 4 - x$

$$y = 4 - 5 = -1$$

Co-ordinates of B are (5,-1)
Area of the shaded region OABC = area under parabola + area of ΔAED + area of ΔDCB

$$= \int_0^1 3x^2 dx + \frac{1}{2} \times 3 \times 3 + \frac{1}{2} \times 1 \times 1$$

$$= \left[x^3\right]_0^1 + 4\frac{1}{2} + \frac{1}{2}$$

$$= 1 + 5$$

= 6 square units

Alternatively
Area of shaded region OABC

$$= \int_0^1 3x^2 dx + \int_1^4 (4 - x)\, dx + \left| \int_4^5 (4 - x)\, dx \right|$$

$$= \left[x^3\right]_0^1 + \left[4x - \frac{x^2}{2}\right]_1^4 + \left| \left[4x - \frac{x^2}{2}\right]_4^5 \right|$$

$$= 1 + 4\frac{1}{2} + \left|\frac{-1}{2}\right|$$

= 6 square units

QUESTION SEVEN

(a) (i) $y = 3 \ln x + 2$

$$\frac{dy}{dx} = \frac{3}{x}, \quad x = 1$$

$$\frac{dy}{dx} = 3$$

If $x = 1$ then $y = 3 \ln 1 + 2 = 2$

The equation of the tangent is given by

$$y - y_1 = m(x - x_1)$$

$$y - 2 = 3(x - 1)$$

i.e. $3x - y - 1 = 0$

(ii) The gradient of the tangent at $x = 1$ is 3. thus the gradient of the normal at $x = 1$ is $-\frac{1}{3}$

(b) $E = 100 + \left(\frac{40}{1 + t}\right)^2$

(i) $t = 0$, $E = 100 + \left(\frac{40}{1 + 0}\right)^2 = 1700$

Rate of emission is 1700 tonnes per year

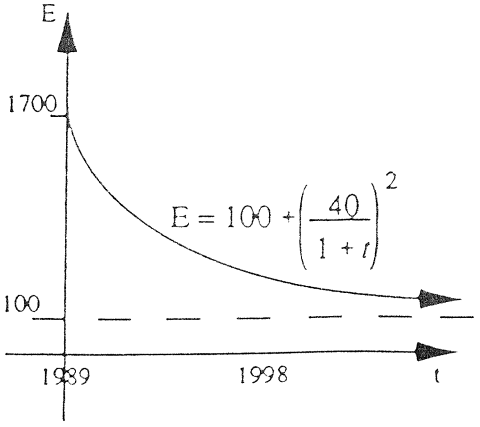
(ii) $t = 9$, $E = 100 + \left(\frac{40}{1 + 9}\right)^2 = 116$

Rate of emission is 116 tonnes per year

(iii) As years pass by t approaches ∞

Thus $\left(\frac{40}{1 + t}\right)^2 \to 0$. Hence $E \to 100$

(iv) Diagram



(v) Total amount of CFCs

$$= \int_0^9 \left(100 + \left(\frac{40}{1+t}\right)^2\right) dt$$

$$= \int_0^9 \left(100 + 1600(1+t)^{-2}\right) dt$$

$$= \left[100t + 1600\frac{(1+t)^{-1}}{-1}\right]_0^9$$

$$= \left[100t - \frac{1600}{1+t}\right]_0^9$$

$$= 900 - 160 - (0 - 1600)$$

$$= 2340 \text{ tonnes}$$

QUESTION EIGHT

(a) $2\cos x = \sqrt{3} \qquad 0 \le x \le 2\pi$

$$\cos x = \frac{\sqrt{3}}{2}$$

The cosine ratio is positive in the first and fourth quadrants

$$x = \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$$

$$= \frac{\pi}{6}, \frac{11\pi}{6}$$

(b) $f(x) = e^{2x}(1-x) \qquad -3 \le x \le 1$

(i) $x = -1 \quad f(-1) = e^{-2}(1 - -1) \approx 0.27$

$x = 0 \quad f(0) = e^0(1-0) = 1.00$

$x = 1 \quad f(1) = e^2(1-1) = 0$

x	-3	-2	-1	0	1
$f(x)$	0.01	0.05	0.27	1.00	0

(ii) $f'(x) = e^{2x}.(-1) + (1-x).2e^{2x}$

by the product rule

$$= e^{2x}(1 - 2x)$$

At a stationary point, $f'(x) = 0$

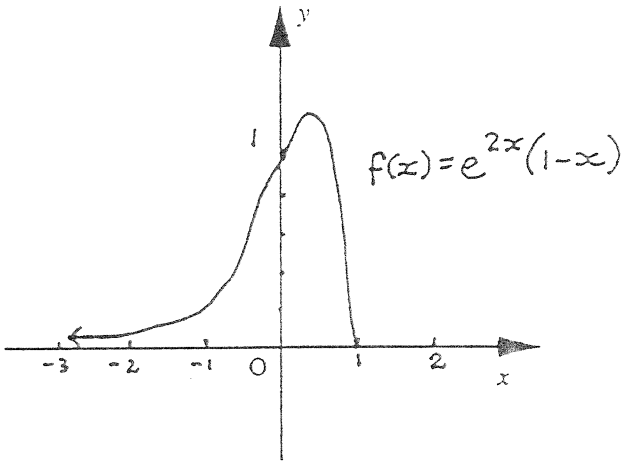
$$\therefore e^{2x}(1-2x) = 0$$

$e^{2x} = 0$ has no solution so $1 - 2x = 0$

$$x = \frac{1}{2}$$

Hence the function has only one stationary point $(\frac{1}{2}, \frac{e}{2})$

(iii),(iv)



(v) Trapezoidal Rule

$$\int_a^b f(x)dx \approx \frac{(b-a)}{2}\left[f(a) + f(b)\right]$$

Required Area =

$$\int_{-3}^{-2} f(x)dx + \int_{-2}^{-1} f(x)d + \int_{-1}^{0} f(x)dx + \int_0^1 f(x)dx$$

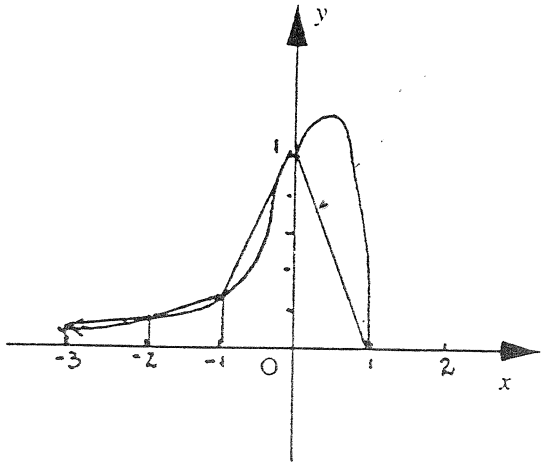
$$\approx \frac{-2+3}{2}\left[0.01 + 0.05\right] + \frac{-1+2}{2}\left[0.05 + 0.27\right]$$

$$+ \frac{0+1}{2}\left[0.27 + 1\right] + \frac{1+0}{2}\left[1 + 0\right]$$

$$\approx \frac{1}{2}\left[0.01 + 2 \times 0.05 + 2 \times 0.27 + 2 \times 1 + 0\right]$$

≈ 1.325 square units

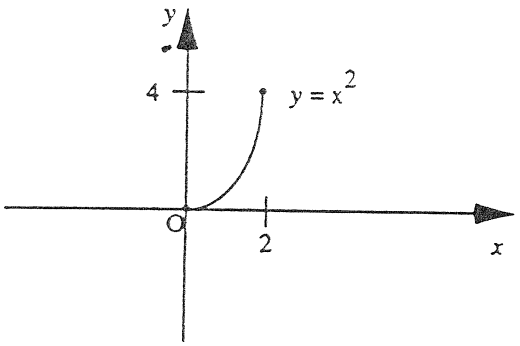
(vi)



Draw 4 trapezia onto the diagram. It is clear from the diagram in the interval $0 \le x \le 1$, the Trapezoidal Rule underestimates the area thus the approximation in 5 is an underestimate.

QUESTION NINE

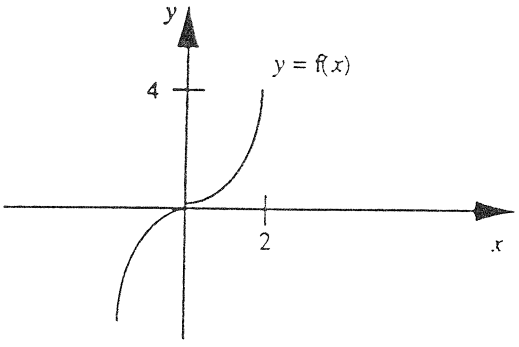
(i)



Domain : $0 \le x \le 2$

Range : $0 \le y \le 4$

(ii) Odd functions have point symmetry about the origin. Thus to complete the sketch, rotate the curve about O.



(iii) $\int_{-2}^{2} f(x)dx = 0$

This is because it is an odd function integrated over an interval symmetric about the origin.

Alternatively

For an odd function $f(-x) = -f(x)$. Thus in the interval $-2 \le x \le 0$ the function is $y = -x^2$

$$\therefore \int_{-2}^{2} f(x)dx = \int_{-2}^{0} (-x)^2 dx + \int_{0}^{2} x^2 dx = 0$$

(b) $\dfrac{dy}{dx} = x^2(2x - 1)(x - 1)$

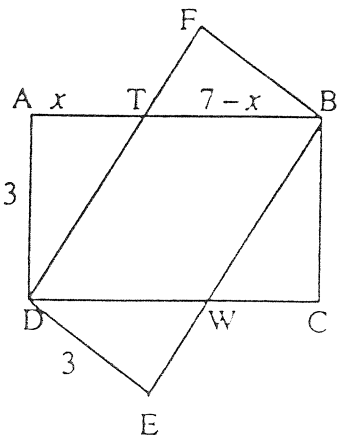
Test for nature of stationary point at $x = 0$

at $x = 0 - \varepsilon$, $\dfrac{dy}{dx} > 0$

at $x = 0 + \varepsilon$, $\dfrac{dy}{dx} > 0$

i.e. at $x = 0$ there is a horizontal point of inflexion.

(c) (i)



Let AT = x units

TB = $(7 - x)$ units

ΔATD is congruent to ΔFTB

[$\angle A = \angle E$ (right angles)

$\angle ATD = \angle FTB$ (vertically opp.)

AD = FB

$\therefore$ By AAS test $\Delta ATD \equiv \Delta FTB$]

$TB = DT = (7 - x)$ units

By Pythagoras' Theorem in ΔATD

$$3^2 + x^2 = (7 - x)^2$$

$$9 + x^2 = 49 - 14x + x^2$$

$$14x = 40$$

$$x = \frac{20}{7}$$

(ii) Area DWBT =

Area ABCD – (Area ΔATD + Area ΔWBC)

$$= 3 \times 7 - (2 \times \frac{1}{2} + 3 \times \frac{20}{7})$$

$$= 12\frac{3}{7} \text{ square units}$$

Alternatively

Area DWBT = 2× Area ΔDTW

$$= 2 \times \frac{1}{2} \times (7 - \frac{20}{7}) \times 3$$

$$= 12\frac{3}{7} \text{ square units}$$

Alternatively

Area DWBT = Area of parallelogram

$$= 3 \times \frac{29}{7}$$

$$= 12\frac{3}{7} \text{ square units}$$

QUESTION TEN

(i) P(win in 3 consecutive draws)

$$= \frac{1}{50} \times \frac{1}{50} \times \frac{1}{50}$$

$$= \frac{1}{125000}$$

$$= .000\ 008$$

(ii) P(win the prize in 1 draw) $= \frac{1}{50}$

$= 0.02$

P(not win the prize in 1 draw) $= 1 - 0.02$

$= 0.98$

P(not win the prize in n draws) $= (0.98)^n$

Either:

For a win to be 99% certain "not to win" must be no more than 1% certain

$$(0.98)^n \leq 0.01$$

$$n \log 0.98 \leq \log 0.01$$

$$n \geq \frac{\log 0.01}{\log 0.98}$$

$$\geq 227.948$$

i.e. at least 228 consecutive draws must be made for it to be 99% certain that the jackpot prize will be won.

Or :

P(win in n consecutive draws) $= 1 - (0.98)^n$

This win is to be 99% certain.

$1 - (0.98)^n \geq 0.99$

$-(0.98)^n \geq -0.01$

$(0.98)^n \leq 0.01$ and proceed as above

(b)

(i) Perimeter of ABCX

= radius AB + arc BC + semi-circle CXA

$$P = (r + r\theta + \frac{1}{2}\pi r) \text{ units} \quad \ldots\ldots(1)$$

(ii) Area of cross-section

= area of sector ABC + area of semi-circle CXA

$$1 = \frac{1}{2} r^2\theta + \frac{1}{2}\pi\left(\frac{r}{2}\right)^2 \quad \ldots\ldots(2)$$

Make θ the subject of (2)

$$1 - \frac{\pi r^2}{8} = \frac{1}{2} r^2\theta \quad \text{(multiply by 2)}$$

$$2 - \frac{\pi r^2}{4} = r^2\theta \quad \text{(divide by } r^2\text{)}$$

$$\theta = \frac{2}{r^2} - \frac{\pi}{4} \quad \ldots\ldots\ldots (3)$$

Substitute (3) into (1)

$$P = r + r\left(\frac{2}{r^2} - \frac{\pi}{4}\right) + \frac{1}{2}\pi r$$

$$= r + \frac{2}{r} - \frac{\pi r}{4} + \frac{\pi r}{2}$$

$$= \frac{2}{r} + r + \frac{\pi r}{4}$$

$$= \frac{2}{r} + r\left(1 + \frac{\pi}{4}\right) \text{ as required}$$

(iii) $\frac{dP}{dr} = \frac{-2}{r^2} + \left(1 + \frac{\pi}{4}\right)$

At stationary point $\frac{dP}{dr} = 0$

$$\frac{-2}{r^2} + 1 + \frac{\pi}{4} = 0$$

$$\frac{4 + \pi}{4} = \frac{2}{r^2}$$

$$\frac{4}{4 + \pi} = \frac{r^2}{2}$$

$$r^2 = \frac{8}{4 + \pi}$$

$\frac{d^2P}{dr^2} = \frac{4}{r^3} > 0$ for all lengths of r (i.e. $r > 0$)

Thus the stationary point is a local minimum
i.e. The least perimeter occurs when

$$r^2 = \frac{8}{\pi + 4}$$

Substitute $r^2 = \frac{8}{\pi + 4}$ into (3)

$$\theta = \frac{2(\pi + 4)}{8} - \frac{\pi}{4}$$

$$= \frac{\pi + 4}{4} - \frac{\pi}{4}$$

$$= \frac{\pi}{4} + 1 - \frac{\pi}{4}$$

$= 1$ radian

$= 57^\circ$ to the nearest degree